

# Operational Calculus and Related Topics

**H.-J. Glaeske**

*Friedrich-Schiller University  
Jena, Germany*

**A.P. Prudnikov**

*(Deceased)*

**K.A. Skòrnik**

*Institute of Mathematics  
Polish Academy of Sciences  
Katowice, Poland*



**Chapman & Hall/CRC**

Taylor & Francis Group

Boca Raton London New York

---

Chapman & Hall/CRC is an imprint of the  
Taylor & Francis Group, an informa business

---

# Contents

<b>Preface</b>	<b>xi</b>
<b>List of Symbols</b>	<b>xv</b>
<b>1 Integral Transforms</b>	<b>1</b>
1.1 Introduction to Operational Calculus . . . . .	1
1.2 Integral Transforms – Introductory Remarks . . . . .	5
1.3 The Fourier Transform . . . . .	8
1.3.1 Definition and Basic Properties . . . . .	8
1.3.2 Examples . . . . .	11
1.3.3 Operational Properties . . . . .	13
1.3.4 The Inversion Formula . . . . .	17
1.3.5 Applications . . . . .	21
1.4 The Laplace Transform . . . . .	27
1.4.1 Definition and Basic Properties . . . . .	27
1.4.2 Examples . . . . .	31
1.4.3 Operational Properties . . . . .	33
1.4.4 The Complex Inversion Formula . . . . .	37
1.4.5 Inversion Methods . . . . .	40
1.4.6 Asymptotic Behavior . . . . .	44
1.4.7 Remarks on the Bilateral Laplace Transform . . . . .	47
1.4.8 Applications . . . . .	49
1.5 The Mellin Transform . . . . .	55
1.5.1 Definition and Basic Properties . . . . .	55
1.5.2 Operational Properties . . . . .	58
1.5.3 The Complex Inversion Formula . . . . .	62
1.5.4 Applications . . . . .	63
1.6 The Stieltjes Transform . . . . .	67
1.6.1 Definition and Basic Properties . . . . .	67
1.6.2 Operational Properties . . . . .	70
1.6.3 Asymptotics . . . . .	73
1.6.4 Inversion and Application . . . . .	75

1.7	The Hilbert Transform . . . . .	78
1.7.1	Definition and Basic Properties . . . . .	78
1.7.2	Operational Properties . . . . .	81
1.7.3	Applications . . . . .	83
1.8	Bessel Transforms . . . . .	84
1.8.1	The Hankel Transform . . . . .	85
1.8.2	The Meijer (K-) Transform . . . . .	93
1.8.3	The Kontorovich–Lebedev Transform . . . . .	100
1.8.4	Application . . . . .	106
1.9	The Mehler–Fock Transform . . . . .	107
1.10	Finite Integral Transforms . . . . .	115
1.10.1	Introduction . . . . .	115
1.10.2	The Chebyshev Transform . . . . .	116
1.10.3	The Legendre Transform . . . . .	122
1.10.4	The Gegenbauer Transform . . . . .	131
1.10.5	The Jacobi Transform . . . . .	137
1.10.6	The Laguerre Transform . . . . .	144
1.10.7	The Hermite Transform . . . . .	153
<b>2</b>	<b>Operational Calculus</b>	<b>163</b>
2.1	Introduction . . . . .	163
2.2	Titchmarsh’s Theorem . . . . .	167
2.3	Operators . . . . .	180
2.3.1	Ring of Functions . . . . .	180
2.3.2	The Field of Operators . . . . .	185
2.3.3	Finite Parts of Divergent Integrals . . . . .	190
2.3.4	Rational Operators . . . . .	201
2.3.5	Laplace Transformable Operators . . . . .	205
2.3.6	Examples . . . . .	213
2.3.7	Periodic Functions . . . . .	217
2.4	Bases of the Operator Analysis . . . . .	219
2.4.1	Sequences and Series of Operators . . . . .	219
2.4.2	Operator Functions . . . . .	226
2.4.3	The Derivative of an Operator Function . . . . .	229
2.4.4	Properties of the Continuous Derivative of an Operator Function . . . . .	229
2.4.5	The Integral of an Operator Function . . . . .	232
2.5	Operators Reducible to Functions . . . . .	236
2.5.1	Regular Operators . . . . .	236

2.5.2	The Realization of Some Operators . . . . .	239
2.5.3	Efros Transforms . . . . .	242
2.6	Application of Operational Calculus . . . . .	247
2.6.1	Ordinary Differential Equations . . . . .	247
2.6.2	Partial Differential Equations . . . . .	258
<b>3</b>	<b>Generalized Functions</b> . . . . .	<b>271</b>
3.1	Introduction . . . . .	271
3.2	Generalized Functions — Functional Approach . . . . .	272
3.2.1	Introduction . . . . .	272
3.2.2	Distributions of One Variable . . . . .	274
3.2.3	Distributional Convergence . . . . .	279
3.2.4	Algebraic Operations on Distributions . . . . .	280
3.3	Generalized Functions — Sequential Approach . . . . .	287
3.3.1	The Identification Principle . . . . .	287
3.3.2	Fundamental Sequences . . . . .	289
3.3.3	Definition of Distributions . . . . .	297
3.3.4	Operations with Distributions . . . . .	300
3.3.5	Regular Operations . . . . .	303
3.4	Delta Sequences . . . . .	311
3.4.1	Definition and Properties . . . . .	311
3.4.2	Distributions as a Generalization of Continuous Functions . . . . .	317
3.4.3	Distributions as a Generalization of Locally Integrable Functions . . . . .	320
3.4.4	Remarks about Distributional Derivatives . . . . .	322
3.4.5	Functions with Poles . . . . .	325
3.4.6	Applications . . . . .	326
3.5	Convergent Sequences . . . . .	332
3.5.1	Sequences of Distributions . . . . .	332
3.5.2	Convergence and Regular Operations . . . . .	339
3.5.3	Distributionally Convergent Sequences of Smooth Functions . . . . .	341
3.5.4	Convolution of Distribution with a Smooth Function of Bounded Support . . . . .	344
3.5.5	Applications . . . . .	346
3.6	Local Properties . . . . .	347
3.6.1	Inner Product of Two Functions . . . . .	347
3.6.2	Distributions of Finite Order . . . . .	350
3.6.3	The Value of a Distribution at a Point . . . . .	352
3.6.4	The Value of a Distribution at Infinity . . . . .	355

3.6.5	Support of a Distribution . . . . .	355
3.7	Irregular Operations . . . . .	355
3.7.1	Definition . . . . .	355
3.7.2	The Integral of Distributions . . . . .	357
3.7.3	Convolution of Distributions . . . . .	369
3.7.4	Multiplication of Distributions . . . . .	375
3.7.5	Applications . . . . .	379
3.8	Hilbert Transform and Multiplication Forms . . . . .	381
3.8.1	Definition of the Hilbert Transform . . . . .	381
3.8.2	Applications and Examples . . . . .	383
	<b>References</b>	<b>389</b>
	<b>Index</b>	<b>401</b>