

# Applied Combinatorics

SECOND EDITION

FRED S. ROBERTS  
BARRY TESMAN



**CRC Press**

Taylor & Francis Group

Boca Raton London New York

---

CRC Press is an imprint of the  
Taylor & Francis Group an **informa** business

A CHAPMAN & HALL BOOK

# Contents

<b>Preface</b>	<b>xvii</b>
<b>Notation</b>	<b>xxvii</b>
<b>1 What Is Combinatorics?</b>	<b>1</b>
1.1 The Three Problems of Combinatorics . . . . .	1
1.2 The History and Applications of Combinatorics . . . . .	8
References for Chapter 1 . . . . .	13
<b>PART I The Basic Tools of Combinatorics</b>	<b>15</b>
<b>2 Basic Counting Rules</b>	<b>15</b>
2.1 The Product Rule . . . . .	15
2.2 The Sum Rule . . . . .	23
2.3 Permutations . . . . .	25
2.4 Complexity of Computation . . . . .	27
2.5 $r$ -Permutations . . . . .	32
2.6 Subsets . . . . .	34
2.7 $r$ -Combinations . . . . .	35
2.8 Probability . . . . .	41
2.9 Sampling with Replacement . . . . .	47
2.10 Occupancy Problems . . . . .	51
2.10.1 The Types of Occupancy Problems . . . . .	51
2.10.2 Case 1: Distinguishable Balls and Distinguishable Cells . . . . .	53
2.10.3 Case 2: Indistinguishable Balls and Distinguishable Cells . . . . .	53
2.10.4 Case 3: Distinguishable Balls and Indistinguishable Cells . . . . .	54
2.10.5 Case 4: Indistinguishable Balls and Indistinguishable Cells . . . . .	55
2.10.6 Examples . . . . .	56
2.11 Multinomial Coefficients . . . . .	59
2.11.1 Occupancy Problems with a Specified Distribution . . . . .	59
2.11.2 Permutations with Classes of Indistinguishable Objects . . . . .	62
2.12 Complete Digest by Enzymes . . . . .	64

2.13	Permutations with Classes of Indistinguishable Objects Revisited . . .	68
2.14	The Binomial Expansion . . . . .	70
2.15	Power in Simple Games . . . . .	73
2.15.1	Examples of Simple Games . . . . .	73
2.15.2	The Shapley-Shubik Power Index . . . . .	75
2.15.3	The U.N. Security Council . . . . .	78
2.15.4	Bicameral Legislatures . . . . .	78
2.15.5	Cost Allocation . . . . .	79
2.15.6	Characteristic Functions . . . . .	80
2.16	Generating Permutations and Combinations . . . . .	84
2.16.1	An Algorithm for Generating Permutations . . . . .	84
2.16.2	An Algorithm for Generating Subsets of Sets . . . . .	86
2.16.3	An Algorithm for Generating Combinations . . . . .	88
2.17	Inversion Distance Between Permutations and the Study of Mutations . . . . .	91
2.18	Good Algorithms . . . . .	96
2.18.1	Asymptotic Analysis . . . . .	96
2.18.2	NP-Complete Problems . . . . .	99
2.19	Pigeonhole Principle and Its Generalizations . . . . .	101
2.19.1	The Simplest Version of the Pigeonhole Principle . . . . .	101
2.19.2	Generalizations and Applications of the Pigeonhole Principle . . . . .	103
2.19.3	Ramsey Numbers . . . . .	106
	Additional Exercises for Chapter 2 . . . . .	111
	References for Chapter 2 . . . . .	113
<b>3</b>	<b>Introduction to Graph Theory</b> . . . . .	<b>119</b>
3.1	Fundamental Concepts . . . . .	119
3.1.1	Some Examples . . . . .	119
3.1.2	Definition of Digraph and Graph . . . . .	124
3.1.3	Labeled Digraphs and the Isomorphism Problem . . . . .	127
3.2	Connectedness . . . . .	133
3.2.1	Reaching in Digraphs . . . . .	133
3.2.2	Joining in Graphs . . . . .	135
3.2.3	Strongly Connected Digraphs and Connected Graphs . . . . .	135
3.2.4	Subgraphs . . . . .	137
3.2.5	Connected Components . . . . .	138
3.3	Graph Coloring and Its Applications . . . . .	145
3.3.1	Some Applications . . . . .	145
3.3.2	Planar Graphs . . . . .	151
3.3.3	Calculating the Chromatic Number . . . . .	154
3.3.4	2-Colorable Graphs . . . . .	155

3.3.5	Graph-Coloring Variants . . . . .	159
3.4	Chromatic Polynomials . . . . .	172
3.4.1	Definitions and Examples . . . . .	172
3.4.2	Reduction Theorems . . . . .	175
3.4.3	Properties of Chromatic Polynomials . . . . .	179
3.5	Trees . . . . .	185
3.5.1	Definition of a Tree and Examples . . . . .	185
3.5.2	Properties of Trees . . . . .	188
3.5.3	Proof of Theorem 3.15 . . . . .	188
3.5.4	Spanning Trees . . . . .	189
3.5.5	Proof of Theorem 3.16 and a Related Result . . . . .	192
3.5.6	Chemical Bonds and the Number of Trees . . . . .	193
3.5.7	Phylogenetic Tree Reconstruction . . . . .	196
3.6	Applications of Rooted Trees to Searching, Sorting, and Phylogeny Reconstruction . . . . .	202
3.6.1	Definitions . . . . .	202
3.6.2	Search Trees . . . . .	205
3.6.3	Proof of Theorem 3.24 . . . . .	206
3.6.4	Sorting . . . . .	207
3.6.5	The Perfect Phylogeny Problem . . . . .	211
3.7	Representing a Graph in the Computer . . . . .	219
3.8	Ramsey Numbers Revisited . . . . .	224
	References for Chapter 3 . . . . .	228
<b>4</b>	<b>Relations</b> . . . . .	<b>235</b>
4.1	Relations . . . . .	235
4.1.1	Binary Relations . . . . .	235
4.1.2	Properties of Relations/Patterns in Digraphs . . . . .	240
4.2	Order Relations and Their Variants . . . . .	247
4.2.1	Defining the Concept of Order Relation . . . . .	247
4.2.2	The Diagram of an Order Relation . . . . .	250
4.2.3	Linear Orders . . . . .	252
4.2.4	Weak Orders . . . . .	254
4.2.5	Stable Marriages . . . . .	256
4.3	Linear Extensions of Partial Orders . . . . .	260
4.3.1	Linear Extensions and Dimension . . . . .	260
4.3.2	Chains and Antichains . . . . .	265
4.3.3	Interval Orders . . . . .	270
4.4	Lattices and Boolean Algebras . . . . .	274
4.4.1	Lattices . . . . .	274
4.4.2	Boolean Algebras . . . . .	276
	References for Chapter 4 . . . . .	282

<b>PART II The Counting Problem</b>	<b>285</b>
<b>5 Generating Functions and Their Applications</b>	<b>285</b>
5.1 Examples of Generating Functions . . . . .	285
5.1.1 Power Series . . . . .	286
5.1.2 Generating Functions . . . . .	288
5.2 Operating on Generating Functions . . . . .	297
5.3 Applications to Counting . . . . .	302
5.3.1 Sampling Problems . . . . .	302
5.3.2 A Comment on Occupancy Problems . . . . .	309
5.4 The Binomial Theorem . . . . .	312
5.5 Exponential Generating Functions and Generating Functions for Permutations . . . . .	320
5.5.1 Definition of Exponential Generating Function . . . . .	320
5.5.2 Applications to Counting Permutations . . . . .	321
5.5.3 Distributions of Distinguishable Balls into Indistinguishable Cells . . . . .	325
5.6 Probability Generating Functions . . . . .	328
5.7 The Coleman and Banzhaf Power Indices . . . . .	333
References for Chapter 5 . . . . .	337
<b>6 Recurrence Relations</b>	<b>339</b>
6.1 Some Examples . . . . .	339
6.1.1 Some Simple Recurrences . . . . .	339
6.1.2 Fibonacci Numbers and Their Applications . . . . .	346
6.1.3 Derangements . . . . .	350
6.1.4 Recurrences Involving More than One Sequence . . . . .	354
6.2 The Method of Characteristic Roots . . . . .	360
6.2.1 The Case of Distinct Roots . . . . .	360
6.2.2 Computation of the $k$ th Fibonacci Number . . . . .	363
6.2.3 The Case of Multiple Roots . . . . .	364
6.3 Solving Recurrences Using Generating Functions . . . . .	369
6.3.1 The Method . . . . .	369
6.3.2 Derangements . . . . .	375
6.3.3 Simultaneous Equations for Generating Functions . . . . .	377
6.4 Some Recurrences Involving Convolutions . . . . .	382
6.4.1 The Number of Simple, Ordered, Rooted Trees . . . . .	382
6.4.2 The Ways to Multiply a Sequence of Numbers in a Computer . . . . .	386
6.4.3 Secondary Structure in RNA . . . . .	389

6.4.4	Organic Compounds Built Up from Benzene Rings . . . . .	391
	References for Chapter 6 . . . . .	400
<b>7</b>	<b>The Principle of Inclusion and Exclusion</b>	<b>403</b>
7.1	The Principle and Some of Its Applications . . . . .	403
7.1.1	Some Simple Examples . . . . .	403
7.1.2	Proof of Theorem 6.1 . . . . .	406
7.1.3	Prime Numbers, Cryptography, and Sieves . . . . .	407
7.1.4	The Probabilistic Case . . . . .	412
7.1.5	The Occupancy Problem with Distinguishable Balls and Cells . . . . .	413
7.1.6	Chromatic Polynomials . . . . .	414
7.1.7	Derangements . . . . .	417
7.1.8	Counting Combinations . . . . .	418
7.1.9	Rook Polynomials . . . . .	419
7.2	The Number of Objects Having Exactly $m$ Properties . . . . .	425
7.2.1	The Main Result and Its Applications . . . . .	425
7.2.2	Proofs of Theorems 7.4 and 7.5 . . . . .	431
	References for Chapter 7 . . . . .	436
<b>8</b>	<b>The Pólya Theory of Counting</b>	<b>439</b>
8.1	Equivalence Relations . . . . .	439
8.1.1	Distinct Configurations and Databases . . . . .	439
8.1.2	Definition of Equivalence Relations . . . . .	440
8.1.3	Equivalence Classes . . . . .	445
8.2	Permutation Groups . . . . .	449
8.2.1	Definition of a Permutation Group . . . . .	449
8.2.2	The Equivalence Relation Induced by a Permutation Group . . . . .	452
8.2.3	Automorphisms of Graphs . . . . .	453
8.3	Burnside's Lemma . . . . .	457
8.3.1	Statement of Burnside's Lemma . . . . .	457
8.3.2	Proof of Burnside's Lemma . . . . .	459
8.4	Distinct Colorings . . . . .	462
8.4.1	Definition of a Coloring . . . . .	462
8.4.2	Equivalent Colorings . . . . .	464
8.4.3	Graph Colorings Equivalent under Automorphisms . . . . .	466
8.4.4	The Case of Switching Functions . . . . .	467
8.5	The Cycle Index . . . . .	472
8.5.1	Permutations as Products of Cycles . . . . .	472
8.5.2	A Special Case of Pólya's Theorem . . . . .	474
8.5.3	Graph Colorings Equivalent under Automorphisms Revisited . . . . .	475

8.5.4	The Case of Switching Functions . . . . .	476
8.5.5	The Cycle Index of a Permutation Group . . . . .	476
8.5.6	Proof of Theorem 8.6 . . . . .	477
8.6	Pólya's Theorem . . . . .	480
8.6.1	The Inventory of Colorings . . . . .	480
8.6.2	Computing the Pattern Inventory . . . . .	482
8.6.3	The Case of Switching Functions . . . . .	484
8.6.4	Proof of Pólya's Theorem . . . . .	485
	References for Chapter 8 . . . . .	488
<b>PART III The Existence Problem</b>		<b>489</b>
<b>9</b>	<b>Combinatorial Designs</b>	<b>489</b>
9.1	Block Designs . . . . .	489
9.2	Latin Squares . . . . .	494
9.2.1	Some Examples . . . . .	494
9.2.2	Orthogonal Latin Squares . . . . .	497
9.2.3	Existence Results for Orthogonal Families . . . . .	500
9.2.4	Proof of Theorem 9.3 . . . . .	505
9.2.5	Orthogonal Arrays with Applications to Cryptography . . . . .	506
9.3	Finite Fields and Complete Orthogonal Families of Latin Squares . . . . .	513
9.3.1	Modular Arithmetic . . . . .	513
9.3.2	Modular Arithmetic and the RSA Cryptosystem . . . . .	514
9.3.3	The Finite Fields $GF(p^k)$ . . . . .	516
9.3.4	Construction of a Complete Orthogonal Family of $n \times n$ Latin Squares if $n$ Is a Power of a Prime . . . . .	519
9.3.5	Justification of the Construction of a Complete Orthogonal Family if $n = p^k$ . . . . .	521
9.4	Balanced Incomplete Block Designs . . . . .	525
9.4.1	$(b, v, r, k, \lambda)$ -Designs . . . . .	525
9.4.2	Necessary Conditions for the Existence of $(b, v, r, k, \lambda)$ -Designs . . . . .	528
9.4.3	Proof of Fisher's Inequality . . . . .	530
9.4.4	Resolvable Designs . . . . .	532
9.4.5	Steiner Triple Systems . . . . .	533
9.4.6	Symmetric Balanced Incomplete Block Designs . . . . .	536
9.4.7	Building New $(b, v, r, k, \lambda)$ -Designs from Existing Ones . . . . .	537
9.4.8	Group Testing and Its Applications . . . . .	539
9.4.9	Steiner Systems and the National Lottery . . . . .	542
9.5	Finite Projective Planes . . . . .	549
9.5.1	Basic Properties . . . . .	549

9.5.2	Projective Planes, Latin Squares, and $(v, k, \lambda)$ -Designs . . . .	553
	References for Chapter 9 . . . . .	558
<b>10</b>	<b>Coding Theory</b>	<b>561</b>
10.1	Information Transmission . . . . .	561
10.2	Encoding and Decoding . . . . .	562
10.3	Error-Correcting Codes . . . . .	567
10.3.1	Error Correction and Hamming Distance . . . . .	567
10.3.2	The Hamming Bound . . . . .	570
10.3.3	The Probability of Error . . . . .	571
10.3.4	Consensus Decoding and Its Connection to Finding Patterns in Molecular Sequences . . . . .	573
10.4	Linear Codes . . . . .	582
10.4.1	Generator Matrices . . . . .	582
10.4.2	Error Correction Using Linear Codes . . . . .	584
10.4.3	Hamming Codes . . . . .	587
10.5	The Use of Block Designs to Find Error-Correcting Codes . . . . .	591
10.5.1	Hadamard Codes . . . . .	591
10.5.2	Constructing Hadamard Designs . . . . .	592
10.5.3	The Richest $(n, d)$ -Codes . . . . .	597
10.5.4	Some Applications . . . . .	602
	References for Chapter 10 . . . . .	605
<b>11</b>	<b>Existence Problems in Graph Theory</b>	<b>609</b>
11.1	Depth-First Search: A Test for Connectedness . . . . .	610
11.1.1	Depth-First Search . . . . .	610
11.1.2	The Computational Complexity of Depth-First Search . . . .	612
11.1.3	A Formal Statement of the Algorithm . . . . .	612
11.1.4	Testing for Connectedness of Truly Massive Graphs . . . .	613
11.2	The One-Way Street Problem . . . . .	616
11.2.1	Robbins' Theorem . . . . .	616
11.2.2	A Depth-First Search Algorithm . . . . .	619
11.2.3	Efficient One-Way Street Assignments . . . . .	621
11.2.4	Efficient One-Way Street Assignments for Grids . . . . .	623
11.2.5	Annular Cities and Communications in Interconnection Networks . . . . .	625
11.3	Eulerian Chains and Paths . . . . .	632
11.3.1	The Königsberg Bridge Problem . . . . .	632
11.3.2	An Algorithm for Finding an Eulerian Closed Chain . . . . .	633
11.3.3	Further Results about Eulerian Chains and Paths . . . . .	635
11.4	Applications of Eulerian Chains and Paths . . . . .	640
11.4.1	The "Chinese Postman" Problem . . . . .	640



11.4.2	Computer Graph Plotting . . . . .	642
11.4.3	Street Sweeping . . . . .	642
11.4.4	Finding Unknown RNA/DNA Chains . . . . .	645
11.4.5	A Coding Application . . . . .	648
11.4.6	De Bruijn Sequences and Telecommunications . . . . .	650
11.5	Hamiltonian Chains and Paths . . . . .	656
11.5.1	Definitions . . . . .	656
11.5.2	Sufficient Conditions for the Existence of a Hamiltonian Circuit in a Graph . . . . .	658
11.5.3	Sufficient Conditions for the Existence of a Hamiltonian Cycle in a Digraph . . . . .	660
11.6	Applications of Hamiltonian Chains and Paths . . . . .	666
11.6.1	Tournaments . . . . .	666
11.6.2	Topological Sorting . . . . .	669
11.6.3	Scheduling Problems in Operations Research . . . . .	670
11.6.4	Facilities Design . . . . .	671
11.6.5	Sequencing by Hybridization . . . . .	673
	References for Chapter 11 . . . . .	678

## **PART IV Combinatorial Optimization** **683**

<b>12</b>	<b>Matching and Covering</b>	<b>683</b>
12.1	Some Matching Problems . . . . .	683
12.2	Some Existence Results: Bipartite Matching and Systems of Distinct Representatives . . . . .	690
12.2.1	Bipartite Matching . . . . .	690
12.2.2	Systems of Distinct Representatives . . . . .	692
12.3	The Existence of Perfect Matchings for Arbitrary Graphs . . . . .	699
12.4	Maximum Matchings and Minimum Coverings . . . . .	702
12.4.1	Vertex Coverings . . . . .	702
12.4.2	Edge Coverings . . . . .	704
12.5	Finding a Maximum Matching . . . . .	706
12.5.1	$M$ -Augmenting Chains . . . . .	706
12.5.2	Proof of Theorem 12.7 . . . . .	707
12.5.3	An Algorithm for Finding a Maximum Matching . . . . .	709
12.6	Matching as Many Elements of $X$ as Possible . . . . .	714
12.7	Maximum-Weight Matching . . . . .	716
12.7.1	The "Chinese Postman" Problem Revisited . . . . .	717
12.7.2	An Algorithm for the Optimal Assignment Problem (Maximum-Weight Matching) . . . . .	718
12.8	Stable Matchings . . . . .	724
12.8.1	Gale-Shapley Algorithm . . . . .	726

12.8.2	Numbers of Stable Matchings . . . . .	727
12.8.3	Structure of Stable Matchings . . . . .	729
12.8.4	Stable Marriage Extensions . . . . .	731
	References for Chapter 12 . . . . .	735
<b>13</b>	<b>Optimization Problems for Graphs and Networks</b>	<b>737</b>
13.1	Minimum Spanning Trees . . . . .	737
13.1.1	Kruskal's Algorithm . . . . .	737
13.1.2	Proof of Theorem 13.1 . . . . .	740
13.1.3	Prim's Algorithm . . . . .	741
13.2	The Shortest Route Problem . . . . .	745
13.2.1	The Problem . . . . .	745
13.2.2	Dijkstra's Algorithm . . . . .	748
13.2.3	Applications to Scheduling Problems . . . . .	751
13.3	Network Flows . . . . .	757
13.3.1	The Maximum-Flow Problem . . . . .	757
13.3.2	Cuts . . . . .	760
13.3.3	A Faulty Max-Flow Algorithm . . . . .	763
13.3.4	Augmenting Chains . . . . .	764
13.3.5	The Max-Flow Algorithm . . . . .	768
13.3.6	A Labeling Procedure for Finding Augmenting Chains . . . . .	770
13.3.7	Complexity of the Max-Flow Algorithm . . . . .	772
13.3.8	Matching Revisited . . . . .	773
13.3.9	Menger's Theorems . . . . .	776
13.4	Minimum-Cost Flow Problems . . . . .	785
13.4.1	Some Examples . . . . .	785
	References for Chapter 13 . . . . .	792
	<b>Appendix: Answers to Selected Exercises</b>	<b>797</b>
	<b>Author Index</b>	<b>833</b>
	<b>Subject Index</b>	<b>841</b>