

C.W. Gardiner P. Zoller

Quantum Noise

A Handbook
of Markovian and Non-Markovian
Quantum Stochastic Methods
with Applications to Quantum Optics

Second Enlarged Edition
With 57 Figures



Springer

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The first edition appeared as Springer Series in Synergetics, Volume 56 under the title:
C.W. Gardiner, *Quantum Noise*

Library of Congress Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek – CIP Einheitsaufnahme

Gardiner, Crispin W.: Quantum noise : a handbook of Markovian and non-Markovian
quantum stochastic methods with applications to quantum optics /

C.W. Gardiner ; P. Zoller. – 2. enl. ed. –

Berlin; Heidelberg; New York; Barcelona; Hong Kong; London; Milan, Paris; Singapore; Tokyo:
Springer, 2000

(Springer series in synergetics : Vol. 56)

ISBN 3-540-66571-4

ISSN 0172-7389

ISBN 3-540-66571-4 2nd Edition Springer-Verlag Berlin Heidelberg New York

ISBN 3-540-53608-6 1st Edition Springer-Verlag Berlin Heidelberg New York

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Printed in Germany

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Typesetting by the authors

Cover design: *design & production*, Heidelberg

Printed on acid-free paper

SPIN: 10744517 55/3144/ba - 5 4 3 2 1 0

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