

UNIVERSITY LECTURE SERIES VOLUME 65

Functions with Disconnected Spectrum

Sampling, Interpolation, Translates

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