

# **Handbook in Monte Carlo Simulation**

Applications in Financial Engineering,  
Risk Management, and Economics

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