

Contents

Part I Grassmann Variables and Applications

| | | |
|----------|---|----|
| 1 | Introduction | 3 |
| 1.1 | History | 3 |
| 1.2 | Applications | 4 |
| | References | 5 |
| 2 | Grassmann Algebra | 7 |
| 2.1 | Elements of the Algebra | 7 |
| 2.2 | Even and Odd Elements, Graded Algebra | 8 |
| 2.3 | Body and Soul, Functions | 10 |
| 2.4 | Exterior Algebra I | 10 |
| | References | 12 |
| 3 | Grassmann Analysis | 13 |
| 3.1 | Differentiation | 13 |
| 3.2 | Integration | 15 |
| 3.3 | Gauss Integrals I | 16 |
| 3.4 | Exterior Algebra II | 21 |
| | References | 27 |
| 4 | Disordered Systems | 29 |
| 4.1 | Introduction | 29 |
| 4.2 | Replica Trick | 30 |
| 4.2.1 | First Variant | 30 |
| 4.2.2 | Second Variant | 30 |
| 4.3 | Quantum Mechanical Particle in a Random Potential | 31 |
| 4.4 | Semicircle Law | 32 |
| | References | 35 |

| | | |
|----------|--|----|
| 5 | Substitution of Variables, Gauss Integrals II | 37 |
| 5.1 | Gauss Integrals II, Pfaffian Form | 37 |
| 5.2 | Variable Substitution I | 38 |
| 5.3 | Gauss Integrals III, Pfaffian Form and Determinant | 41 |
| 6 | The Complex Conjugate | 45 |
| 6.1 | Description | 45 |
| 6.2 | Similarity to Antilinear Operations in Quantum Mechanics | 46 |
| | Reference | 46 |
| 7 | Path Integrals for Fermions and Bosons | 47 |
| 7.1 | Coherent States | 47 |
| 7.2 | Path Integral Representation | 49 |
| 7.3 | Free Particles | 53 |
| | 7.3.1 Starting from Functions of τ | 53 |
| | 7.3.2 Matsubara Frequencies | 54 |
| 7.4 | Interacting Systems and Feynman Diagrams | 57 |
| | References | 65 |
| 8 | Dimers in Two Dimensions | 67 |
| 8.1 | General Considerations | 67 |
| 8.2 | Square Lattice | 69 |
| 8.3 | Dimers and Tilings | 71 |
| | References | 73 |
| 9 | Two-Dimensional Ising Model | 75 |
| 9.1 | The Ising Model | 75 |
| | 9.1.1 The Model | 75 |
| | 9.1.2 Phases and Singularities | 76 |
| 9.2 | Representation by Grassmann Variables | 78 |
| 9.3 | Evaluation of the Partition Function | 81 |
| 9.4 | Loops Winding Around the Torus | 82 |
| 9.5 | Divergence of the Specific Heat | 84 |
| 9.6 | Other Lattices | 85 |
| 9.7 | Phases and Boundary Tension | 86 |
| | 9.7.1 Appendix | 89 |
| 9.8 | Duality Transformation | 93 |
| | 9.8.1 Order and Disorder Operators | 95 |
| | References | 99 |

Part II Supermathematics

| | | |
|-----------|---|-----|
| 10 | Supermatrices | 103 |
| 10.1 | Differential, Matrices, Transposition | 103 |
| 10.2 | Chain Rule, Matrix Multiplication | 105 |
| 10.3 | Berezinian Superdeterminant | 106 |
| 10.4 | Supertrace and Differential of Superdeterminant | 109 |

| | | |
|-----------|--|------------|
| 10.5 | Parity Transposition | 110 |
| | References | 111 |
| 11 | Functions of Matrices | 113 |
| 11.1 | The Inverse | 113 |
| 11.2 | Analytic Functions | 114 |
| 12 | Supersymmetric Matrices | 117 |
| 12.1 | Quadratic Form | 117 |
| 12.2 | Gauss Integrals IV, Superpfaffian, Expectation Values | 118 |
| 12.3 | Orthosymplectic Transformation and Group | 120 |
| 13 | Adjoint, Scalar Product, Superunitary Groups | 123 |
| 13.1 | Adjoint | 123 |
| 13.1.1 | Adjoint of the First Kind | 123 |
| 13.1.2 | Adjoint of the Second Kind | 124 |
| 13.1.3 | Adjoint and Transposition: Summary | 124 |
| 13.2 | Scalar Product, Superunitary Group | 125 |
| 13.2.1 | First Kind | 125 |
| 13.2.2 | Second Kind | 126 |
| 13.3 | Gauss Integrals V | 127 |
| 14 | Superreal Matrices, Unitary-Orthosymplectic Groups | 131 |
| 14.1 | Matrices and Groups for the Adjoint of Second Kind | 131 |
| 14.2 | Vector Products | 133 |
| 14.3 | Gauss Integrals VI, Superreal Vectors | 134 |
| 15 | Integral Theorems for the Unitary Group | 139 |
| 15.1 | Integral Theorem for Functions of Vectors Invariant Under Superunitary Groups | 139 |
| 15.1.1 | Introduction | 139 |
| 15.1.2 | Theorem for Superunitary Vectors of First Kind | 140 |
| 15.1.3 | Proof of the Theorem for $N = 1$ | 141 |
| 15.1.4 | Generalization to Natural N | 142 |
| 15.1.5 | Consequences | 143 |
| 15.2 | Integral Theorem for Quasihermitian Matrices: Superunitary Group | 143 |
| 15.2.1 | Introduction and Theorem, ‘Quasihermitian’ | 143 |
| 15.2.2 | Integral Theorem for One Matrix $\in \mathcal{M}(1, 1)$ | 145 |
| 15.2.3 | Integral Theorem for N Matrices $Q \in \mathcal{M}(1, 1)$ | 148 |
| 15.2.4 | Integral Theorem for N Matrices $Q \in \mathcal{M}(n, m)$ | 149 |
| 15.2.5 | Final Remarks | 150 |
| 15.3 | Matrix as a Set of Vectors | 151 |
| | References | 153 |

| | | |
|--|--|-----|
| 16 | Integral Theorems for the (Unitary-)Orthosymplectic Group | 155 |
| 16.1 | Integral Theorem for Vectors | 155 |
| 16.1.1 | Invariance Under the Orthosymplectic Group..... | 155 |
| 16.1.2 | Invariance Under the Unitary-Orthosymplectic Group | 157 |
| 16.2 | Integral Theorem for Quasihermitian and Quasireal Matrices: Invariance Under UOSp | 159 |
| 16.2.1 | Theorem | 160 |
| 16.2.2 | Invariant Function $f(Q)$, $Q \in \mathcal{M}(2, 2)$ | 161 |
| 16.2.3 | The Integral for $N = 1$, $Q \in \mathcal{M}(2, 2)$ | 164 |
| 16.2.4 | The General Case | 165 |
| 16.3 | Integral Theorem for Quasiantiermitian Quasireal Matrices..... | 165 |
| 16.3.1 | The Theorem | 165 |
| 16.3.2 | Invariant Function $f(Q)$, $Q \in \mathcal{M}(2, 2)$ | 166 |
| 16.3.3 | The Integral for $N = 1$, $Q \in \mathcal{M}(2, 2)$ | 168 |
| 16.3.4 | The General Case | 169 |
| 16.3.5 | Matrix as a Set of Vectors | 169 |
| 17 | More on Matrices | 171 |
| 17.1 | Eigenvalue Problem | 171 |
| 17.2 | Diagonalization of Superreal Hermitian Matrices | 174 |
| 17.3 | Functional Equation for Matrices..... | 176 |
| 17.4 | Berezinian for Transformation of Matrices with Linearly Dependent Matrix Elements | 178 |
| Part III Supersymmetry in Statistical Physics | | |
| 18 | Supersymmetric Models | 183 |
| 18.1 | Supersymmetric Quantum Mechanics..... | 183 |
| 18.1.1 | Supersymmetric Partners | 183 |
| 18.1.2 | Harmonic Oscillator | 185 |
| 18.1.3 | The \cosh^{-2} -Potential | 185 |
| 18.1.4 | Supersymmetric δ -Potential | 186 |
| 18.1.5 | Hydrogen Spectrum | 187 |
| 18.2 | Chiral and Supersymmetric Models with $Q^2 = 0$ | 188 |
| 18.2.1 | Chiral Models | 188 |
| 18.2.2 | Fermions on a Lattice | 189 |
| | References..... | 190 |
| 19 | Supersymmetry in Stochastic Field Equations and in High Energy Physics | 193 |
| 19.1 | Stochastic Time-Dependent Equations | 193 |
| 19.1.1 | Langevin and Fokker-Planck Equation | 193 |
| 19.1.2 | Time-Dependent Correlation Functions | 195 |
| 19.1.3 | Supersymmetry and Fluctuation-Dissipation Theorem ... | 196 |

| | | |
|-----------|--|------------|
| 19.2 | Supersymmetry in High Energy Physics | 199 |
| | References | 201 |
| 20 | Dimensional Reduction | 203 |
| 20.1 | Rotational Invariance in Superreal Space | 203 |
| 20.1.1 | Lie Superalgebra and Jacobi Identity | 203 |
| 20.1.2 | Unitary-Orthosymplectic Rotations and Supersymmetric Laplace Operator | 204 |
| 20.2 | Ising Model in a Stochastic Magnetic Field | 206 |
| 20.3 | Branched Polymers and Lattice Animals | 210 |
| 20.4 | Electron in the Lowest Landau Level | 212 |
| 20.4.1 | Free Electron in a Magnetic Field | 212 |
| 20.4.2 | Random Potential | 213 |
| 20.4.3 | Supersymmetric Lagrangian | 216 |
| 20.4.4 | Dimensional Reduction | 218 |
| 20.5 | Isotropic $\phi^{2\sigma}$ -Theories with Negative Number of Components ... | 221 |
| | References | 222 |
| 21 | Random Matrix Theory | 227 |
| 21.1 | Green's Functions | 227 |
| 21.2 | Reduction of the Gaussian Unitary Ensemble to a Matrix Model | 228 |
| 21.3 | Saddle Point | 231 |
| 21.4 | Convergence and Symmetry | 234 |
| 21.5 | Nonlinear σ -Model | 237 |
| 21.5.1 | Efetov Parametrization | 237 |
| 21.5.2 | Invariant Measure | 238 |
| 21.5.3 | Singularity of the Invariant Measure | 240 |
| 21.5.4 | Schäfer-Wegner Parametrization | 241 |
| 21.5.5 | Pruisken-Schäfer Parametrization | 243 |
| 21.5.6 | The Nonlinear σ -Model Finally | 245 |
| 21.6 | Green's Functions | 245 |
| 21.7 | Gaussian Orthogonal and Symplectic Ensembles | 248 |
| 21.7.1 | Gaussian Orthogonal Ensemble | 248 |
| 21.7.2 | Gaussian Symplectic Ensemble | 250 |
| 21.8 | Circular Ensembles and Level Distributions | 252 |
| 21.8.1 | Circular Ensembles | 252 |
| 21.8.2 | Level Distribution | 253 |
| 21.9 | Final Remarks | 255 |
| | References | 257 |
| 22 | Diffusive Model | 261 |
| 22.1 | Correlation Functions | 261 |
| 22.1.1 | Equilibrium Correlations | 261 |
| 22.1.2 | Linear Response | 262 |

| | | |
|------------------|---|------------|
| 22.2 | The Unitary Model: Green's Functions and Action | 263 |
| 22.3 | Saddle Point and First Order | 266 |
| 22.4 | Second Order and Fluctuations | 269 |
| | 22.4.1 Diffusion | 271 |
| | 22.4.2 Conductivity | 273 |
| 22.5 | Nonlinear σ -Model | 274 |
| 22.6 | Orthogonal Case | 280 |
| | 22.6.1 The Lattice Model | 280 |
| | 22.6.2 Saddle Point and Fluctuations, Cooperon | 283 |
| 22.7 | Symplectic Case | 286 |
| | 22.7.1 The Lattice Model | 286 |
| | 22.7.2 Saddle Point and Fluctuations | 292 |
| | 22.7.3 Some Simplifications | 299 |
| | 22.7.4 The Extreme and Pure Case | 299 |
| | References | 301 |
| 23 | More on the Non-linear σ-Model | 303 |
| | 23.1 Beyond the Saddle-Point Solution | 303 |
| | 23.1.1 Symmetry and Correlations | 304 |
| | 23.1.2 Scaling Theory of Conductivity | 309 |
| | 23.1.3 Density Fluctuations and Multifractality | 313 |
| | 23.2 Ten Symmetry Classes | 316 |
| | 23.2.1 Wigner-Dyson Classes | 316 |
| | 23.2.2 Chiral Classes | 317 |
| | 23.2.3 Bogolubov-de Gennes Classes | 318 |
| | 23.2.4 Summary | 319 |
| | 23.2.5 Topological Insulators and Superconductors | 320 |
| | 23.3 More in Two Dimensions | 321 |
| | 23.3.1 Integer Quantum Hall Effect | 321 |
| | 23.3.2 Spin Quantum Hall Effect | 322 |
| | 23.3.3 Quantum Spin Hall Effect | 322 |
| | 23.3.4 Spin Hall Effect | 322 |
| | 23.3.5 Thermal Quantum Hall Effect | 323 |
| | 23.3.6 Wess-Zumino Term | 323 |
| | 23.3.7 Graphene | 323 |
| | 23.4 Superbosonization | 323 |
| | References | 329 |
| 24 | Summary and Additional Remarks | 335 |
| | References | 338 |
| Solutions | | 341 |
| | Problems of Chap. 2 | 341 |
| | Problems of Chap. 3 | 342 |
| | Problem of Chap. 4 | 343 |
| | Problems of Chap. 5 | 344 |

| | |
|----------------------------|------------|
| Problem of Chap. 6..... | 345 |
| Problems of Chap. 7 | 345 |
| Problems of Chap. 8 | 346 |
| Problems of Chap. 9 | 348 |
| Problem of Chap. 10 | 352 |
| Problems of Chap. 11 | 352 |
| Problems of Chap. 12 | 353 |
| Problems of Chap. 13 | 353 |
| Problems of Chap. 14 | 354 |
| Problem of Chap. 15 | 354 |
| Problems of Chap. 18 | 355 |
| Problems of Chap. 20 | 356 |
| Problems of Chap. 21 | 356 |
| Problems of Chap. 22 | 357 |
| Problem of Chap. 23 | 357 |
| References | 359 |
| Index | 371 |