Contents

Preface xi

Chapter 1. Introduction 1
§1.1. The three classical linear PDEs 1
§1.2. Nonlinear PDEs 4
§1.3. Our choice of equations and the idea of modulation equations 6
§1.4. Overview 11

Part I Nonlinear dynamics in $\mathbb{R}^d$

Chapter 2. Basic ODE dynamics 15
§2.1. Linear systems 17
§2.2. Local existence and uniqueness for nonlinear systems 34
§2.3. Special solutions 38
§2.4. ω-limit sets and attractors 49
§2.5. Chaotic dynamics 58
§2.6. Examples 64

Chapter 3. Dissipative dynamics 75
§3.1. Bifurcations 76
§3.2. Center manifold theory 85
§3.3. The Hopf bifurcation 91
§3.4. Routes to chaos 98

Chapter 4. Hamiltonian dynamics 109

§4.1. Basic properties 109
§4.2. Some celestial mechanics 116
§4.3. Completely integrable systems 121
§4.4. Perturbations of completely integrable systems 123
§4.5. Homoclinic chaos 128

Part II Nonlinear dynamics in countably many dimensions

Chapter 5. PDEs on an interval 133
§5.1. From finitely to infinitely many dimensions 134
§5.2. Basic function spaces and Fourier series 151
§5.3. The Chafee-Infante problem 167

Chapter 6. The Navier-Stokes equations 179
§6.1. Introduction 179
§6.2. The equations on a torus 186
§6.3. Other boundary conditions and more general domains 197

Part III PDEs on the infinite line

Chapter 7. Some dissipative PDE models 205
§7.1. The KPP equation 206
§7.2. The Allen-Cahn equation 222
§7.3. Intermezzo: Fourier transform 225
§7.4. The Burgers equation 237

Chapter 8. Three canonical modulation equations 249
§8.1. The NLS equation 250
§8.2. The KdV equation 259
§8.3. The GL equation 275

Chapter 9. Reaction-Diffusion systems 295
§9.1. Modeling. and existence and uniqueness 297
§9.2. Two classical examples 302
§9.3. The Turing instability 307

Part IV Modulation theory and applications

Chapter 10. Dynamics of pattern and the GL equation 315
§10.1. Introduction 316

§10.2. The Swift-Hohenberg equation 319
§10.3. The universality of the GL equation 332
§10.4. An abstract approximation result 337
§10.5. Reaction-Diffusion systems 347
§10.6. Convection problems 354
§10.7. The Couette-Taylor problem 370
§10.8. Attractors for pattern forming systems 378
§10.9. Further remarks 395

Chapter 11. Wave packets and the NLS equation 401
§11.1. Introduction 402
§11.2. Justification in case of cubic nonlinearities 404
§11.3. The universality of the NLS equation 411
§11.4. Quadratic nonlinearities 416
§11.5. Extension of the theory 421
§11.6. Pulse dynamics in photonic crystals 429
§11.7. Nonlinear optics 440

Chapter 12. Long waves and their modulation equations 451
§12.1. An approximation result 452
§12.2. The universality of the KdV equation 456
§12.3. Whitham, Boussinesq, BBM, etc. 465
§12.4. The long wave limit 468

Chapter 13. Center manifold reduction and spatial dynamics 473
§13.1. The center manifold theorem 473
§13.2. Local bifurcation theory on bounded domains 478
§13.3. Spatial dynamics for elliptic problems in a strip 482
§13.4. Applications 484

Chapter 14. Diffusive stability 497
§14.1. Linear and nonlinear diffusive behavior 498
§14.2. Diffusive stability of spatially periodic equilibria 507
§14.3. The critical case 523
§14.4. Phase diffusion equations 529
§14.5. Dispersive dynamics 535

Bibliography 541

List of symbols 567

Index 569