

# Contents

	page
Introduction . . . . .	XVII

## Chapter One

### Operator theory in finite-dimensional vector spaces

§ 1. Vector spaces and normed vector spaces . . . . .	1
1. Basic notions . . . . .	1
2. Bases . . . . .	2
3. Linear manifolds . . . . .	3
4. Convergence and norms . . . . .	4
5. Topological notions in a normed space . . . . .	6
6. Infinite series of vectors . . . . .	7
7. Vector-valued functions . . . . .	8
§ 2. Linear forms and the adjoint space . . . . .	10
1. Linear forms . . . . .	10
2. The adjoint space . . . . .	11
3. The adjoint basis . . . . .	12
4. The adjoint space of a normed space . . . . .	13
5. The convexity of balls . . . . .	14
6. The second adjoint space . . . . .	15
§ 3. Linear operators . . . . .	16
1. Definitions. Matrix representations . . . . .	16
2. Linear operations on operators . . . . .	18
3. The algebra of linear operators . . . . .	19
4. Projections. Nilpotents . . . . .	20
5. Invariance. Decomposition . . . . .	22
6. The adjoint operator . . . . .	23
§ 4. Analysis with operators . . . . .	25
1. Convergence and norms for operators . . . . .	25
2. The norm of $T^n$ . . . . .	27
3. Examples of norms . . . . .	28
4. Infinite series of operators . . . . .	29
5. Operator-valued functions . . . . .	31
6. Pairs of projections . . . . .	32
§ 5. The eigenvalue problem . . . . .	34
1. Definitions . . . . .	34
2. The resolvent . . . . .	36
3. Singularities of the resolvent . . . . .	38
4. The canonical form of an operator . . . . .	40
5. The adjoint problem . . . . .	43
6. Functions of an operator . . . . .	44
7. Similarity transformations . . . . .	46

§ 6. Operators in unitary spaces . . . . .	47
1. Unitary spaces . . . . .	47
2. The adjoint space . . . . .	48
3. Orthonormal families . . . . .	49
4. Linear operators . . . . .	51
5. Symmetric forms and symmetric operators . . . . .	52
6. Unitary, isometric and normal operators . . . . .	54
7. Projections . . . . .	55
8. Pairs of projections . . . . .	56
9. The eigenvalue problem . . . . .	58
10. The minimax principle . . . . .	60

## Chapter Two

### Perturbation theory in a finite-dimensional space 62

§ 1. Analytic perturbation of eigenvalues . . . . .	63
1. The problem . . . . .	63
2. Singularities of the eigenvalues . . . . .	65
3. Perturbation of the resolvent . . . . .	66
4. Perturbation of the eigenprojections . . . . .	67
5. Singularities of the eigenprojections . . . . .	69
6. Remarks and examples . . . . .	70
7. The case of $T(\varkappa)$ linear in $\varkappa$ . . . . .	72
8. Summary . . . . .	73
§ 2. Perturbation series . . . . .	74
1. The total projection for the $\lambda$ -group . . . . .	74
2. The weighted mean of eigenvalues . . . . .	77
3. The reduction process . . . . .	81
4. Formulas for higher approximations . . . . .	83
5. A theorem of MOTZKIN-TAUSSKY . . . . .	85
6. The ranks of the coefficients of the perturbation series . . . . .	86
§ 3. Convergence radii and error estimates . . . . .	88
1. Simple estimates . . . . .	88
2. The method of majorizing series . . . . .	89
3. Estimates on eigenvectors . . . . .	91
4. Further error estimates . . . . .	93
5. The special case of a normal unperturbed operator . . . . .	94
6. The enumerative method . . . . .	97
§ 4. Similarity transformations of the eigenspaces and eigenvectors . . . . .	98
1. Eigenvectors . . . . .	98
2. Transformation functions . . . . .	99
3. Solution of the differential equation . . . . .	102
4. The transformation function and the reduction process . . . . .	104
5. Simultaneous transformation for several projections . . . . .	104
6. Diagonalization of a holomorphic matrix function . . . . .	106
§ 5. Non-analytic perturbations . . . . .	106
1. Continuity of the eigenvalues and the total projection . . . . .	106
2. The numbering of the eigenvalues . . . . .	108
3. Continuity of the eigenspaces and eigenvectors . . . . .	110
4. Differentiability at a point . . . . .	111

5. Differentiability in an interval . . . . .	113
6. Asymptotic expansion of the eigenvalues and eigenvectors . . . . .	115
7. Operators depending on several parameters . . . . .	116
8. The eigenvalues as functions of the operator . . . . .	117
§ 6. Perturbation of symmetric operators . . . . .	120
1. Analytic perturbation of symmetric operators . . . . .	120
2. Orthonormal families of eigenvectors . . . . .	121
3. Continuity and differentiability . . . . .	122
4. The eigenvalues as functions of the symmetric operator . . . . .	124
5. Applications. A theorem of LIDSKII . . . . .	124

### Chapter Three

#### Introduction to the theory of operators in Banach spaces

§ 1. Banach spaces . . . . .	127
1. Normed spaces . . . . .	127
2. Banach spaces . . . . .	129
3. Linear forms . . . . .	132
4. The adjoint space . . . . .	134
5. The principle of uniform boundedness . . . . .	136
6. Weak convergence . . . . .	137
7. Weak* convergence . . . . .	140
8. The quotient space . . . . .	140
§ 2. Linear operators in Banach spaces . . . . .	142
1. Linear operators. The domain and range . . . . .	142
2. Continuity and boundedness . . . . .	145
3. Ordinary differential operators of second order. . . . .	146
§ 3. Bounded operators . . . . .	149
1. The space of bounded operators . . . . .	149
2. The operator algebra $\mathcal{B}(X)$ . . . . .	153
3. The adjoint operator . . . . .	154
4. Projections . . . . .	155
§ 4. Compact operators . . . . .	157
1. Definition . . . . .	157
2. The space of compact operators . . . . .	158
3. Degenerate operators. The trace and determinant . . . . .	160
§ 5. Closed operators . . . . .	163
1. Remarks on unbounded operators . . . . .	163
2. Closed operators . . . . .	164
3. Closable operators . . . . .	165
4. The closed graph theorem . . . . .	166
5. The adjoint operator . . . . .	167
6. Commutativity and decomposition . . . . .	171
§ 6. Resolvents and spectra . . . . .	172
1. Definitions . . . . .	172
2. The spectra of bounded operators . . . . .	176
3. The point at infinity . . . . .	176
4. Separation of the spectrum . . . . .	178

5. Isolated eigenvalues . . . . .	180
6. The resolvent of the adjoint . . . . .	183
7. The spectra of compact operators . . . . .	185
8. Operators with compact resolvent . . . . .	187

## Chapter Four

### Stability theorems

§ 1. Stability of closedness and bounded invertibility . . . . .	189
1. Stability of closedness under relatively bounded perturbation . . . . .	189
2. Examples of relative boundedness . . . . .	191
3. Relative compactness and a stability theorem . . . . .	194
4. Stability of bounded invertibility . . . . .	196
§ 2. Generalized convergence of closed operators . . . . .	197
1. The gap between subspaces . . . . .	197
2. The gap and the dimension . . . . .	199
3. Duality . . . . .	200
4. The gap between closed operators . . . . .	201
5. Further results on the stability of bounded invertibility . . . . .	205
6. Generalized convergence . . . . .	206
§ 3. Perturbation of the spectrum . . . . .	208
1. Upper semicontinuity of the spectrum . . . . .	208
2. Lower semi-discontinuity of the spectrum . . . . .	209
3. Continuity and analyticity of the resolvent . . . . .	210
4. Semicontinuity of separated parts of the spectrum . . . . .	212
5. Continuity of a finite system of eigenvalues . . . . .	213
6. Change of the spectrum under relatively bounded perturbation . . . . .	214
7. Simultaneous consideration of an infinite number of eigenvalues . . . . .	215
8. An application to Banach algebras. Wiener's theorem . . . . .	216
§ 4. Pairs of closed linear manifolds . . . . .	218
1. Definitions . . . . .	218
2. Duality . . . . .	221
3. Regular pairs of closed linear manifolds . . . . .	223
4. The approximate nullity and deficiency . . . . .	225
5. Stability theorems . . . . .	227
§ 5. Stability theorems for semi-Fredholm operators . . . . .	229
1. The nullity, deficiency and index of an operator . . . . .	229
2. The general stability theorem . . . . .	232
3. Other stability theorems . . . . .	236
4. Isolated eigenvalues . . . . .	239
5. Another form of the stability theorem . . . . .	241
6. Structure of the spectrum of a closed operator . . . . .	242
§ 6. Degenerate perturbations . . . . .	244
1. The Weinstein-Aronszajn determinants . . . . .	244
2. The W-A formulas . . . . .	246
3. Proof of the W-A formulas . . . . .	248
4. Conditions excluding the singular case . . . . .	249

Chapter Five  
Operators in Hilbert spaces

§ 1.	Hilbert space . . . . .	251
	1. Basic notions . . . . .	251
	2. Complete orthonormal families . . . . .	254
§ 2.	Bounded operators in Hilbert spaces . . . . .	256
	1. Bounded operators and their adjoints . . . . .	256
	2. Unitary and isometric operators . . . . .	257
	3. Compact operators . . . . .	260
	4. The Schmidt class . . . . .	262
	5. Perturbation of orthonormal families . . . . .	264
§ 3.	Unbounded operators in Hilbert spaces . . . . .	267
	1. General remarks . . . . .	267
	2. The numerical range . . . . .	267
	3. Symmetric operators . . . . .	269
	4. The spectra of symmetric operators . . . . .	270
	5. The resolvents and spectra of selfadjoint operators . . . . .	272
	6. Second-order ordinary differential operators . . . . .	274
	7. The operators $T^*T$ . . . . .	275
	8. Normal operators . . . . .	276
	9. Reduction of symmetric operators . . . . .	277
	10. Semibounded and accretive operators . . . . .	278
	11. The square root of an $m$ -accretive operator . . . . .	281
§ 4.	Perturbation of selfadjoint operators . . . . .	287
	1. Stability of selfadjointness . . . . .	287
	2. The case of relative bound 1 . . . . .	289
	3. Perturbation of the spectrum . . . . .	290
	4. Semibounded operators . . . . .	291
	5. Completeness of the eigenprojections of slightly non-selfadjoint operators . . . . .	293
§ 5.	The Schrödinger and Dirac operators . . . . .	297
	1. Partial differential operators . . . . .	297
	2. The Laplacian in the whole space . . . . .	299
	3. The Schrödinger operator with a static potential . . . . .	302
	4. The Dirac operator . . . . .	305

Chapter Six

Sesquilinear forms in Hilbert spaces and associated operators

§ 1.	Sesquilinear and quadratic forms . . . . .	308
	1. Definitions . . . . .	308
	2. Semiboundedness . . . . .	310
	3. Closed forms . . . . .	313
	4. Closable forms . . . . .	315
	5. Forms constructed from sectorial operators . . . . .	318
	6. Sums of forms . . . . .	319
	7. Relative boundedness for forms and operators . . . . .	321
§ 2.	The representation theorems . . . . .	322
	1. The first representation theorem . . . . .	322
	2. Proof of the first representation theorem . . . . .	323
	3. The Friedrichs extension . . . . .	325
	4. Other examples for the representation theorem . . . . .	326

5. Supplementary remarks . . . . .	328
6. The second representation theorem . . . . .	331
7. The polar decomposition of a closed operator . . . . .	334
§ 3. Perturbation of sesquilinear forms and the associated operators . . . . .	336
1. The real part of an $m$ -sectorial operator . . . . .	336
2. Perturbation of an $m$ -sectorial operator and its resolvent . . . . .	338
3. Symmetric unperturbed operators . . . . .	340
4. Pseudo-Friedrichs extensions . . . . .	341
§ 4. Quadratic forms and the Schrödinger operators . . . . .	343
1. Ordinary differential operators . . . . .	343
2. The Dirichlet form and the Laplace operator . . . . .	346
3. The Schrödinger operators in $\mathbb{R}^3$ . . . . .	348
4. Bounded regions . . . . .	352
§ 5. The spectral theorem and perturbation of spectral families . . . . .	353
1. Spectral families . . . . .	353
2. The selfadjoint operator associated with a spectral family . . . . .	356
3. The spectral theorem . . . . .	360
4. Stability theorems for the spectral family . . . . .	361

## Chapter Seven

### Analytic perturbation theory

§ 1. Analytic families of operators . . . . .	365
1. Analyticity of vector- and operator-valued functions . . . . .	365
2. Analyticity of a family of unbounded operators . . . . .	366
3. Separation of the spectrum and finite systems of eigenvalues . . . . .	368
4. Remarks on infinite systems of eigenvalues . . . . .	371
5. Perturbation series . . . . .	372
6. A holomorphic family related to a degenerate perturbation . . . . .	373
§ 2. Holomorphic families of type (A) . . . . .	375
1. Definition . . . . .	375
2. A criterion for type (A) . . . . .	377
3. Remarks on holomorphic families of type (A) . . . . .	379
4. Convergence radii and error estimates . . . . .	381
5. Normal unperturbed operators . . . . .	383
§ 3. Selfadjoint holomorphic families . . . . .	385
1. General remarks . . . . .	385
2. Continuation of the eigenvalues . . . . .	387
3. The Mathieu, Schrödinger, and Dirac equations . . . . .	389
4. Growth rate of the eigenvalues . . . . .	390
5. Total eigenvalues considered simultaneously . . . . .	392
§ 4. Holomorphic families of type (B) . . . . .	393
1. Bounded-holomorphic families of sesquilinear forms . . . . .	393
2. Holomorphic families of forms of type (a) and holomorphic families of operators of type (B) . . . . .	395
3. A criterion for type (B) . . . . .	398
4. Holomorphic families of type $(B_0)$ . . . . .	401
5. The relationship between holomorphic families of types (A) and (B) . . . . .	403
6. Perturbation series for eigenvalues and eigenprojections . . . . .	404
7. Growth rate of eigenvalues and the total system of eigenvalues . . . . .	407
8. Application to differential operators . . . . .	408
9. The two-electron problem . . . . .	410

§ 5. Further problems of analytic perturbation theory . . . . .	413
1. Holomorphic families of type (C) . . . . .	413
2. Analytic perturbation of the spectral family . . . . .	414
3. Analyticity of $ H(\lambda) $ and $ H(\lambda) ^0$ . . . . .	416
§ 6. Eigenvalue problems in the generalized form . . . . .	416
1. General considerations . . . . .	416
2. Perturbation theory . . . . .	419
3. Holomorphic families of type (A) . . . . .	421
4. Holomorphic families of type (B) . . . . .	422
5. Boundary perturbation . . . . .	423

Chapter Eight

Asymptotic perturbation theory

§ 1. Strong convergence in the generalized sense . . . . .	427
1. Strong convergence of the resolvent . . . . .	427
2. Generalized strong convergence and spectra . . . . .	431
3. Perturbation of eigenvalues and eigenvectors . . . . .	433
4. Stable eigenvalues . . . . .	437
§ 2. Asymptotic expansions . . . . .	439
1. Asymptotic expansion of the resolvent . . . . .	439
2. Remarks on asymptotic expansions . . . . .	442
3. Asymptotic expansions of isolated eigenvalues and eigenvectors . . . . .	443
4. Further asymptotic expansions . . . . .	446
§ 3. Generalized strong convergence of sectorial operators . . . . .	451
1. Convergence of a sequence of bounded forms . . . . .	451
2. Convergence of sectorial forms "from above" . . . . .	453
3. Nonincreasing sequences of symmetric forms . . . . .	457
4. Convergence from below . . . . .	459
5. Spectra of converging operators . . . . .	460
§ 4. Asymptotic expansions for sectorial operators . . . . .	461
1. The problem. The zeroth approximation for the resolvent . . . . .	461
2. The 1/2-order approximation for the resolvent . . . . .	463
3. The first and higher order approximations for the resolvent . . . . .	464
4. Asymptotic expansions for eigenvalues and eigenvectors . . . . .	468
§ 5. Spectral concentration . . . . .	471
1. Unstable eigenvalues . . . . .	471
2. Spectral concentration . . . . .	472
3. Pseudo-eigenvectors and spectral concentration . . . . .	473
4. Asymptotic expansions . . . . .	474

Chapter Nine

Perturbation theory for semigroups of operators

§ 1. One-parameter semigroups and groups of operators . . . . .	477
1. The problem . . . . .	477
2. Definition of the exponential function . . . . .	478
3. Properties of the exponential function . . . . .	480
4. Bounded and quasi-bounded semigroups . . . . .	484
5. Solution of the inhomogeneous differential equation . . . . .	486
6. Holomorphic semigroups . . . . .	487
7. The inhomogeneous differential equation for a holomorphic semi-group . . . . .	491
8. Applications to the heat and Schrödinger equations . . . . .	493

§ 2. Perturbation of semigroups . . . . .	495
1. Analytic perturbation of quasi-bounded semigroups . . . . .	495
2. Analytic perturbation of holomorphic semigroups . . . . .	497
3. Perturbation of contraction semigroups . . . . .	499
4. Convergence of quasi-bounded semigroups in a restricted sense . . . . .	500
5. Strong convergence of quasi-bounded semigroups . . . . .	501
6. Asymptotic perturbation of semigroups . . . . .	504
§ 3. Approximation by discrete semigroups . . . . .	507
1. Discrete semigroups . . . . .	507
2. Approximation of a continuous semigroup by discrete semigroups . . . . .	509
3. Approximation theorems . . . . .	511
4. Variation of the space . . . . .	512

## Chapter Ten

### Perturbation of continuous spectra and unitary equivalence

§ 1. The continuous spectrum of a selfadjoint operator . . . . .	514
1. The point and continuous spectra . . . . .	514
2. The absolutely continuous and singular spectra . . . . .	516
3. The trace class . . . . .	519
4. The trace and determinant . . . . .	521
§ 2. Perturbation of continuous spectra . . . . .	523
1. A theorem of WEYL-VON NEUMANN . . . . .	523
2. A generalization . . . . .	525
§ 3. Wave operators and the stability of absolutely continuous spectra . . . . .	527
1. Introduction . . . . .	527
2. Generalized wave operators . . . . .	529
3. A sufficient condition for the existence of the wave operator . . . . .	533
4. An application to potential scattering . . . . .	534
§ 4. Existence and completeness of wave operators . . . . .	535
1. Perturbations of rank one (special case) . . . . .	535
2. Perturbations of rank one (general case) . . . . .	538
3. Perturbations of the trace class . . . . .	540
4. Wave operators for functions of operators . . . . .	543
5. Strengthening of the existence theorems . . . . .	547
6. Dependence of $W_{\pm}(H_2, H_1)$ on $H_1$ and $H_2$ . . . . .	551
§ 5. A stationary method . . . . .	551
1. Introduction . . . . .	551
2. The $I$ operations . . . . .	553
3. Equivalence with the time-dependent theory . . . . .	555
4. The $I$ operations on degenerate operators . . . . .	556
5. Solution of the integral equation for rank $A = 1$ . . . . .	558
6. Solution of the integral equation for a degenerate $A$ . . . . .	561
7. Application to differential operators . . . . .	563
Bibliography . . . . .	566
Articles . . . . .	566
Books and monographs . . . . .	576
Notation index . . . . .	578
Author index . . . . .	580
Subject index . . . . .	583
Errata . . . . .	591