

# Contents

<b>1</b>	<b>Rotation about a Fixed Point. Reflection in a Plane</b>	<b>1</b>
1.1	Direction Cosine Matrix	1
1.2	Similarity Transformation	6
1.3	Euler Angles	7
1.4	Bryan Angles	9
1.5	Rotation Tensor	11
1.6	Euler-Rodrigues Parameters	16
1.7	Relationships Between Euler-Rodrigues Parameters and Euler Angles	19
1.8	Quaternions	20
1.9	Relationships Between Three Positions of a Body	24
1.10	Relationships Between four Positions	26
1.11	Cayley-Klein Parameters	28
1.12	Euler Vector. Exponential Form of the Direction Cosine Matrix	33
1.13	Rodrigues Vector	34
1.14	Wiener Vector	36
1.15	Illustrative Problems	38
1.15.1	Generalized Coordinates Associated with a Given Direction Cosine Matrix	38
1.15.2	Resultant of two 180°-Rotations	38
1.15.3	Rotations $(n, \varphi)$ Resulting in Positions with the Critical Bryan Angle $\phi_2 = \pm\pi/2$	40
1.15.4	Determine all Direction Cosine Matrices Having three Prescribed Elements	40
1.15.5	Rear Axle of a Vehicle	42
1.15.6	Rotation Determined from Three Positions of a Body-Fixed Point	43
1.15.7	Rodrigues Vector Determined from Prescribed Point Displacements	44
1.15.8	Spherical Interpolation	45

1.15.9	Rotations Effecting a Prescribed Line Displacement . . . . .	46
1.15.10	Sensor Calibration . . . . .	48
1.15.11	Decomposition of a Rotation into three Rotations . . . . .	51
1.15.12	Decomposition of a Rotation into three Rotations. Quaternion Formulation . . . . .	55
1.16	Reflection in a Plane . . . . .	58
	References . . . . .	61
<b>2</b>	<b>Line Geometry . . . . .</b>	<b>63</b>
2.1	Normal Vector of a Plane. Equation of a Plane . . . . .	63
2.2	Plücker Vectors. Plücker Coordinates of a Line . . . . .	64
2.3	Reflection in a Line . . . . .	65
2.4	Plücker Vectors of the Line of Intersection of two Planes . . . . .	66
2.5	Condition for two Lines to Intersect . . . . .	66
2.6	Plücker Vectors of the Common Perpendicular of two Lines . . . . .	69
2.7	Linear Complex . . . . .	70
2.7.1	Null Point. Null Plane . . . . .	70
2.7.2	Axis. Pitch . . . . .	71
2.7.3	Determine the Null Point if the Null Plane is Given and Vice Versa . . . . .	73
2.7.4	Determine a Linear Complex from Given Complex Lines . . . . .	74
2.7.5	Reciprocal Polars . . . . .	74
2.8	Linear Congruence . . . . .	77
2.9	Ruled Surfaces . . . . .	78
2.9.1	Intersection of three Linear Complexes . . . . .	78
2.9.2	Striction Point. Distribution Parameter . . . . .	78
	References . . . . .	83
<b>3</b>	<b>Finite Screw Displacement . . . . .</b>	<b>85</b>
3.1	( $4 \times 4$ ) Transformation Matrix . . . . .	85
3.2	Chasles' Theorem . . . . .	87
3.3	Scalar Measures of a Screw Displacement . . . . .	90
3.4	Roth' Theorem . . . . .	91
3.5	Screw Displacement Determined from Displacements of Three Body Points . . . . .	93
3.6	Halphen's Theorem . . . . .	94
3.7	Resultant of two Screw Displacements. Screw Triangle . . . . .	95
3.8	Dual Numbers . . . . .	97
3.9	Dual Vectors. Dual Angles . . . . .	99
3.10	Principle of Transference . . . . .	102
3.10.1	Dual Basis. Dual Direction Cosine Matrix . . . . .	102
3.10.2	Screw Axis, Screw Angle and Translation Determined from Dual Direction Cosines . . . . .	104
3.10.3	Dual Euler Angles. Dual Bryan Angles . . . . .	108

3.10.4	Dual Rodrigues Vector .....	110
3.10.5	Dual Euler-Rodrigues Parameters. Dual Quaternions ..	110
3.11	Resultant of two Screw Displacements. Dual-Quaternion Formulation .....	112
3.12	Equations for the Screw Triangle .....	118
3.13	Resultant of two Infinitesimal Screw Displacements. Cylindroid .....	122
3.14	Screw Displacements Effecting a Prescribed Line Displacement	127
	References .....	134
<b>4</b>	<b>Degree of Freedom of a Mechanism .....</b>	<b>137</b>
4.1	Grübler's Formula .....	138
4.2	Illustrative Examples .....	140
4.2.1	Five-Point-Contact Joint .....	141
4.2.2	Shaky Truss .....	143
4.2.3	Closed Chain Formed by Four Planar Four-Bars .....	144
4.2.4	Trihedral Plane-Symmetric Bricard Mechanism .....	145
4.2.5	Line-Symmetric Bricard Mechanism .....	149
4.2.6	Homokinetic Shaft Coupling .....	153
4.2.7	Mobile Tilings .....	154
	References .....	156
<b>5</b>	<b>Spatial Simple Closed Chains .....</b>	<b>159</b>
5.1	Joint Variables. Denavit-Hartenberg Parameters .....	161
5.2	Screw Displacements. Coordinate Transformations .....	162
5.3	Closure Conditions .....	166
5.3.1	Woernle-Lee Equations .....	166
5.3.2	Half-Angle Equations .....	175
5.4	Systematic Analysis of Mechanisms .....	177
5.4.1	RCCC .....	177
5.4.2	RCRRC and CRRRC .....	181
5.4.3	RCPRC, CCPRR and RCPCR. Independent Variable in the Prismatic Joint .....	184
5.4.4	Mechanisms in Rows 6 and 7 of Table 5.1. Independent Variable is an Angle .....	184
5.4.5	5R-C .....	185
5.4.6	RRCRPR, RRCPRR, RRCRRP. Independent Variable in the Prismatic Joint .....	188
5.4.7	Mechanism 7R .....	188
5.4.8	4R-3P. Independent Variable is an Angle .....	192
5.4.9	6R-P. Independent Variable is an Angle .....	192
5.4.10	6R-P. Independent Variable in the Prismatic Joint ..	194
5.5	Mechanisms with Special Parameter Values .....	194
5.5.1	7R with Three Parallel Joint Axes in Series .....	194
5.5.2	RRSRR .....	195

5.6	Generalized Velocities. Generalized Accelerations .....	197
5.6.1	RCCC .....	198
5.6.2	Mechanism 7R .....	198
5.7	Spatial Serial Robots .....	199
	References .....	201
<b>6</b>	<b>Overconstrained Mechanisms .....</b>	<b>205</b>
6.1	Bricard's Theorem on Closed Chains with Revolute Joints ..	206
6.2	Bennett Mechanism .....	207
6.3	Kinematical Chains with Five Revolute Joints .....	210
6.3.1	Goldberg Mechanism .....	212
6.4	Kinematical Chains with Six Revolute Joints .....	217
6.4.1	Line-Symmetric Bricard Mechanism .....	217
6.4.2	Plane-Symmetric Bricard Mechanism .....	221
6.4.3	Trihedral Bricard Mechanism .....	223
6.4.4	Dietmaier's Mechanism .....	229
6.5	Mobile Polyhedra .....	231
6.6	RRCRP Mechanism .....	233
6.7	4R-P Mechanism .....	234
6.8	Bricard-Borel Mechanism .....	235
6.9	Hyperboloid and Paraboloid Mechanisms .....	236
6.10	Cam Mechanism .....	240
6.11	Heureka Octahedron .....	242
	References .....	254
<b>7</b>	<b>Two-Joint Chains .....</b>	<b>257</b>
7.1	Work Space of Points of the Terminal Body .....	258
7.1.1	Chains RR Defining Tori .....	259
7.1.2	Chains RP Defining Hyperboloids of Revolution .....	262
7.1.3	Chains PR Defining Elliptic Cylinders .....	262
7.1.4	Chains PP Defining Planes .....	263
7.2	Chains RR Leading the Terminal Body Through Prescribed Positions .....	263
7.3	Chains CC Leading the Terminal Body Through Prescribed Positions .....	271
	References .....	275
<b>8</b>	<b>Stewart Platform .....</b>	<b>277</b>
8.1	Direct Kinematics of the General Stewart Platform .....	278
8.2	Triangle-Configuration of the Stewart Platform .....	284
	References .....	287

<b>9 Angular Velocity. Angular Acceleration</b>	289
9.1 Definitions. Basic Equations	289
9.2 Inverse Motion	292
9.3 Instantaneous Screw Axis. Pitch. Velocity Screw. Linear Complex of Velocity	293
9.4 Angular Velocity of a Body in Terms of Positions and Velocities of Three Points	299
9.5 Raccording Axodes. Striction Point. Distribution Parameter	301
9.6 Spatial Rotation About a Fixed Point	305
9.7 The Ancient Chinese Southpointing Chariot	307
9.8 Acceleration Distribution. Instantaneous Center of Acceleration	309
9.9 Angular Acceleration of a Body in Terms of Positions, Velocities and Accelerations of Three Points	316
9.10 Strapdown Inertial Navigation	318
9.11 Motion on a Curved Surface	321
9.12 Mecanum Wheel	324
References	327
<b>10 Kinematic Differential Equations</b>	329
10.1 Direction Cosines	329
10.2 Euler Angles	333
10.3 Bryan Angles	333
10.4 Euler-Rodrigues Parameters	334
10.5 Cayley-Klein Parameters	337
10.6 Rodrigues Parameters	338
10.7 Wiener Parameters	339
10.8 Euler Vector	339
10.9 Correction Formulas for Euler-Rodrigues Parameters	345
References	348
<b>11 Direct Kinematics of Tree-Structured Systems</b>	349
11.1 Kinematics of Individual Joints	350
11.2 Kinematics of Entire Systems	353
11.2.1 Recursive Solution	354
11.2.2 Explicit Solution	354
References	357
<b>12 Screw Systems</b>	359
12.1 Resultant of two Velocity Screws. Cylindroid	359
12.2 Relative Velocity Screw. Raccording Hyperboloids	362
12.3 Rotary Piercing of Tubes over Plug	365
12.4 Analogy Between Force Screw and Velocity Screw	366
12.5 Virtual Power of a Force Screw. Reciprocal Screws	368
12.6 Reciprocal Screw Systems	372

12.6.1	First-Order Screw System and Reciprocal Fifth-Order System . . . . .	373
12.6.2	Second-Order Screw System and Reciprocal Fourth-Order System . . . . .	374
12.6.3	Third-Order Screw System and Reciprocal Third-Order System . . . . .	381
References	.....	384
<b>13</b>	<b>Shaft Couplings</b> . . . . .	387
13.1	Hooke's Joint . . . . .	388
13.1.1	Polhode and Herpolhode Cones of the Central Cross . . . . .	390
13.2	Fenyi's Joint . . . . .	393
13.2.1	Raccording Axodes of the Central Ring . . . . .	395
13.3	Series-Connected Hooke's Joints . . . . .	397
13.4	Homokinetic Shaft Couplings . . . . .	400
13.4.1	Couplings With a Spherical Joint . . . . .	401
13.4.2	Couplings With Three Parallel Serial Chains . . . . .	403
13.4.3	Ball-in-Track Joints . . . . .	404
13.4.4	Tripod Joint . . . . .	407
References	.....	410
<b>14</b>	<b>Displacements in a Plane</b> . . . . .	411
14.1	Complex Numbers in Planar Kinematics . . . . .	411
14.1.1	Curvature of a Plane Curve . . . . .	414
14.2	Elementary Displacements . . . . .	415
14.3	Resultant Displacements. Commutativity Conditions . . . . .	418
14.4	Relationships Between Three Positions . . . . .	425
14.5	Relationships Between Four Positions. Pole Curve . . . . .	432
14.6	Tilings . . . . .	441
References	.....	448
<b>15</b>	<b>Plane Motion</b> . . . . .	451
15.1	Instantaneous Center of Rotation. Centrodes . . . . .	451
15.1.1	Theorems of Burmester and Kennedy/Aronhold . . . . .	453
15.1.2	Illustrative Examples . . . . .	455
15.2	Velocity and Acceleration in Complex Formulation . . . . .	470
15.2.1	Instantaneous Center of Rotation . . . . .	471
15.2.2	Instantaneous Center of Acceleration . . . . .	472
15.2.3	Inflection Circle. Bresse Circles . . . . .	473
15.2.4	Center of Acceleration and Bresse Circles of the Inverse Motion . . . . .	475
15.3	Curvature of Plane Trajectories . . . . .	476
15.3.1	Normal Poles . . . . .	476
15.3.2	Normal Poles of the Inverse Motion . . . . .	478
15.3.3	Euler-Savary Equation . . . . .	479

15.3.4 Radii of Curvatures of Centrodes . . . . .	485
15.3.5 Cubic of Stationary Curvature. Directrix . . . . .	487
15.3.6 Ball's Point . . . . .	492
15.4 Holditch's Theorem . . . . .	493
15.5 Trochoids . . . . .	495
15.5.1 Basic Equations in Complex Notation . . . . .	497
15.5.2 Double Generation of Trochoids . . . . .	499
15.5.3 Cycloids . . . . .	501
15.5.4 Ordinary Cycloids . . . . .	506
15.5.5 Involute of a Circle . . . . .	507
15.5.6 Dwell Mechanisms Based on Cycloids . . . . .	509
15.6 Rectangle Moving Between two Lines and a Point . . . . .	513
15.6.1 Obtuse-Angled Corner . . . . .	515
15.6.2 Right-Angled Corner . . . . .	520
15.6.3 Acute-Angled Corner . . . . .	521
References . . . . .	527
<b>16 Theory of Gearing . . . . .</b>	<b>529</b>
16.1 Parallel Axes . . . . .	530
16.1.1 Curvature Relationship of Meshing Tooth Flanks . . . . .	532
16.1.2 Camus' Theorem . . . . .	534
16.1.3 Cycloidal Gearing . . . . .	535
16.1.4 Construction of Conjugate Flanks . . . . .	537
16.1.5 Pin Gears . . . . .	540
16.1.6 External Involute Spur Gears . . . . .	543
16.1.7 Internal Involute Spur Gears . . . . .	551
16.1.8 Involute Helical Gearing . . . . .	552
16.2 Skew Axes . . . . .	558
16.2.1 Construction of Conjugate Flanks . . . . .	558
16.2.2 General Spatial Involute Gearing . . . . .	560
References . . . . .	564
<b>17 Planar Four-Bar Mechanism . . . . .</b>	<b>567</b>
17.1 Grashof Condition . . . . .	568
17.2 Transfer Function . . . . .	573
17.3 Interchange of Input Link and Fixed Link . . . . .	576
17.4 Inclination Angle of the Coupler. Transmission Angle . . . . .	577
17.5 Transmission Ratio. Angular Acceleration of Output Link . . . . .	578
17.6 Stationary Values of the Transmission Ratio . . . . .	581
17.7 Transmission of Forces and Torques . . . . .	587
17.8 Coupler Curves . . . . .	590
17.8.1 Roberts/Tschebychev Theorem. Cognate Four-Bars . . . . .	590
17.8.2 Parameter Equations for Coupler Curves . . . . .	594
17.8.3 Implicit Equation for Coupler Curves . . . . .	595
17.8.4 Symmetrical Coupler Curves . . . . .	603

17.9	Slider-Crank. Inverted Slider-Crank . . . . .	606
17.10	Planar Parallel Robot . . . . .	609
17.11	Four-Bars with Prescribed Transmission Characteristics . . . . .	611
17.11.1	Prescribed Pairs of Input-Output Angles . . . . .	611
17.11.2	Prescribed Transmission Ratios . . . . .	613
17.11.3	Jeantaud's Steering Mechanism . . . . .	614
17.12	Coupler Curves with Prescribed Properties . . . . .	616
17.12.1	Coupler Curves Passing Through Prescribed Points . . . . .	616
17.12.2	Straight-Line Approximations . . . . .	617
17.12.3	Tschebychev's Straight-Line Approximations . . . . .	619
17.13	Peaucellier Inversor . . . . .	626
17.14	Four-Bars Producing Prescribed Positions of the Coupler Plane. Burmester Theory . . . . .	629
17.14.1	Three Prescribed Positions . . . . .	629
17.14.2	Four Prescribed Positions. Center Point Curve. Circle Point Curves . . . . .	630
17.14.3	Five Prescribed Positions . . . . .	632
17.14.4	Crank-Rockers Producing Four Prescribed Positions in Prescribed Order . . . . .	632
17.15	Trajectory of the Center of Mass of a Four-Bar . . . . .	634
	References . . . . .	636
<b>18</b>	<b>Spherical Four-Bar Mechanism . . . . .</b>	<b>639</b>
18.1	Transfer Function . . . . .	640
18.2	Grashof Type Conditions . . . . .	641
18.3	Coupler Curves . . . . .	643
18.3.1	Implicit Equation for Coupler Curves . . . . .	644
18.3.2	Symmetrical Coupler Curves . . . . .	646
18.3.3	Geometrical Locus of Double Points . . . . .	648
18.3.4	Stereographic Projection . . . . .	651
18.3.5	Cusps . . . . .	654
18.3.6	Parameter Equations for Coupler Curves . . . . .	656
18.4	Spherical Parallel Robot . . . . .	658
	References . . . . .	662
<b>19</b>	<b>Dynamics of Mechanisms . . . . .</b>	<b>663</b>
19.1	Conservative Single-Degree-of-Freedom Mechanisms . . . . .	663
19.2	The General Problem of Dynamics . . . . .	664
19.3	Principle of Virtual Power . . . . .	665
19.4	Equations of Motion . . . . .	669
19.4.1	Systems with Tree Structure . . . . .	669
19.4.2	Constraint Forces and Torques in Joints . . . . .	671
19.4.3	Systems with Closed Kinematical Chains . . . . .	672
	References . . . . .	675

<b>References to Additional Literature .....</b>	677
<b>Index .....</b>	679