

Contents

Preface — VII

General Scheme of Notations — XIX

1 Introduction — 1

2 Geometry of Physical Space — 3

2.1 Euclidean Affine Space — 4

2.1.1 General Definitions — 4

2.1.2 Topology on \mathcal{E} — 9

2.1.3 Smooth Mappings on \mathcal{E} — 9

2.1.4 Parallel Transport — 13

2.1.5 Total Derivatives of Vector and Tensor Fields — 14

2.1.6 Integration — 16

2.1.7 Symmetrization and Antisymmetrization — 17

2.1.8 Divergence — 17

2.1.9 Curl — 18

2.2 Curvilinear Coordinates — 19

2.2.1 Reparametrization — 19

2.2.2 Coordinate Curves — 21

2.2.3 Local Basis — 21

2.2.4 From Pointwise to the Whole — 22

2.2.5 Metric Tensors — 23

2.2.6 Parallel Transport — 24

2.2.7 Total Derivatives — 24

2.2.8 Integration — 28

2.2.9 The Physical Basis — 28

2.3 Nabla Operator Formalism in 3D-Space — 28

2.3.1 Definitions of Nabla — 29

2.3.2 Operations in Euclidean Space — 29

2.3.3 Nabla and Mappings — 30

2.3.4 Coordinate Representations for Divergence — 32

2.3.5 Coordinate Representations for Curl — 33

2.4 Riemannian Space — 33

2.4.1 General Definition — 33

2.4.2 Smooth Mappings on \mathcal{P} — 35

2.4.3 Geometric Measures and Geodesics — 37

2.4.4 Parallel Transport — 38

2.4.5 Integration — 40

- 2.5 **Newtonian Space-Time — 40**
- 2.5.1 **Newtonian Space-Time Manifold — 41**
- 2.5.2 **Newton's Laws — 42**
- 2.5.3 **Rigid Frame — 44**
- 2.5.4 **Change of Frame — 44**
- 2.6 **Relativistic Space-Time — 46**
- 2.6.1 **Lorentzian Manifolds — 46**
- 2.6.2 **Time Orientation — 47**
- 2.6.3 **Definition of Relativistic Space-Time — 48**
- 2.6.4 **Observers — 48**
- 2.6.5 **Lorentz Transformations — 49**
- 2.6.6 **Matter — 49**
- 2.6.7 **The Einstein Equation — 51**
- 2.7 **Concluding Remarks — 57**

- 3 Essentials of Non-Linear Elasticity Theory — 59**
- 3.1 **Shapes and Deformation — 59**
- 3.2 **Shape Coordinates and Basis — 61**
- 3.3 **Deformation Gradient and Strain Measures — 64**
- 3.3.1 **Deformation Gradient — 64**
- 3.3.2 **Strain Measures — 66**
- 3.3.3 **Deformation Gradient in Curvilinear Coordinates — 67**
- 3.3.4 **Strain Measures in Curvilinear Coordinates — 68**
- 3.4 **Displacement Field — 69**
- 3.5 **Motion — 70**
- 3.6 **Compatibility Conditions — 70**
- 3.6.1 **Review on de Rham Cohomology — 70**
- 3.6.2 **Necessary and Sufficient Conditions for Compatibility — 71**
- 3.7 **Stresses — 75**
- 3.8 **Non-Linear Elasticity as Field Theory — 78**
- 3.8.1 **Action and Its Lagrangian — 78**
- 3.8.2 **Partial and Full Variations — 79**
- 3.8.3 **Field Equations — 84**
- 3.8.4 **Action Invariance Conditions — 84**
- 3.9 **Constitutive Relations — 87**
- 3.9.1 **Principle of Material-Frame Indifference — 87**
- 3.9.2 **The Cauchy Polar Decomposition Theorem — 89**
- 3.9.3 **Simple Material — 90**
- 3.9.4 **Representation Theorems — 92**
- 3.10 **Hyperelastic Solids — 94**
- 3.10.1 **Expressions for Stresses — 94**
- 3.10.2 **Universal Deformations — 94**

- 3.11 **Linearized Elasticity — 103**
- 3.11.1 **Linearized Kinematics — 103**
- 3.11.2 **Linearized Constitutive Relations — 105**
- 3.11.3 **Linearized Stresses — 106**
- 3.11.4 **The Green–Rivlin–Shield–Truesdell Formula — 107**
- 3.12 **Distributed Defects in Solids — 109**
- 3.12.1 **Preliminary Remarks — 109**
- 3.12.2 **Total Distortion 1-Forms — 110**
- 3.12.3 **Four-Dimensional Space and the Homotopy Operator — 111**
- 3.12.4 **Decomposition of Total Distortion into Exact and Antixact Parts — 112**
- 3.12.5 **Continuity Equations of Defect Dynamics — 112**
- 3.12.6 **4D Representation — 113**
- 3.12.7 **The Momentum Equation — 113**
- 3.12.8 **Equations in Matrix Form — 114**
- 3.12.9 **Relations of Dislocation and Disclination Forms with Connection, Curvature, and Torsion Forms — 114**
- 3.12.10 **Application of Yang–Mills Coupling Theory — 116**

- 4 Geometric Formalization of the Body and Its Representation in Physical Space — 119**
- 4.1 **Geometric Motivation — 119**
- 4.2 **Comparison between Conventional and Non-Euclidean Continuum Mechanics — 125**
- 4.3 **A Body — 128**
- 4.4 **Configurations — 132**
- 4.5 **A Shape of a Body as a Submanifold of the Physical Space — 135**
- 4.5.1 **A Shape as a Submanifold — 135**
- 4.5.2 **The Local k -slice Condition — 136**
- 4.5.3 **The Induced Riemannian Space Structure — 137**
- 4.5.4 **A Shape and the Physical Space. Intrinsic versus Spatial — 137**

- 5 Strain Measures — 141**
- 5.1 **Review on Cauchy Theory — 141**
- 5.2 **Configurations and Deformations — 143**
- 5.3 **Coordinate Representations of Configurations and Deformations — 144**
- 5.3.1 **The General Case — 144**
- 5.3.2 **The Euclidean Case — 146**
- 5.4 **Two-Point Tensors — 149**
- 5.4.1 **Two-Point Tensor Bundle — 149**
- 5.4.2 **The Transpose and Orthogonal Tensors — 151**
- 5.5 **Configuration Gradient — 153**
- 5.6 **Left and Right Cauchy–Green Strain Tensors — 156**

- 5.6.1 Spatial Measurements in Material Description — 156
- 5.6.2 The Cauchy Polar Decomposition Theorem — 157
- 5.6.3 Cauchy–Green Strain Measures as Pullback and Pushforward of Metrics — 158

- 6 Motion — 161**
 - 6.1 Motion as a Curve — 161
 - 6.2 Velocity — 162
 - 6.3 $(m + 1)$ -Formalism and Acceleration — 164
 - 6.4 Flows and Lie Derivatives — 165
 - 6.4.1 Vector Fields and Integral Curves — 165
 - 6.4.2 Flow — 166
 - 6.4.3 Lie Derivatives — 168
 - 6.4.4 Time-Dependent Flow — 174
 - 6.5 Motion as a Time-Dependent Flow — 176

- 7 Stress Measures — 179**
 - 7.1 Concentrated Forces and Force Densities — 179
 - 7.2 Inclined Hyperplanes — 181
 - 7.3 Piola Section on Inclined Hyperplanes — 183
 - 7.4 The Cauchy Section — 185
 - 7.5 Cauchy and Piola Stresses — 186
 - 7.6 Transformation from Spatial Description to Material — 188
 - 7.7 Example: Neo-Hookean Solids — 191
 - 7.8 The Case $\dim \mathfrak{B} = \dim \mathfrak{P} = 3$ — 192
 - 7.9 The Eshelby Energy-Momentum Tensor on Manifolds — 193

- 8 Material Uniformity and Inhomogeneity — 195**
 - 8.1 Equivalence Relation Between Smooth Embeddings — 195
 - 8.2 Local Configurations and Simple Bodies — 195
 - 8.3 Material Uniformity — 196
 - 8.4 A Material Metric — 199
 - 8.5 Bodies with Variable Material Composition — 200
 - 8.5.1 Formalization of Bodies with Variable Material Composition — 200
 - 8.5.2 The Discrete Process — 203
 - 8.5.3 The Continuous Process — 204

- 9 Material Connections — 209**
 - 9.1 Connections on Vector Bundles — 209
 - 9.2 Affine Connection — 212
 - 9.2.1 The Transformation Law — 213
 - 9.2.2 Torsion, Curvature, and Non-Metricity — 214

- 9.2.3 A Particular Case: Euclidean Space — 216
- 9.2.4 A Particular Case: Riemannian Space — 218
- 9.2.5 Connection on the Pullback Bundle — 220
- 9.2.6 The Moving Frame Method — 222
- 9.2.7 The Weitzenböck Connection as a Material Connection — 225
- 9.2.8 The Weitzenböck Connection: Example — 226

- 10 Balance Equations — 231**
 - 10.1 Divergence — 231
 - 10.1.1 The Case of a Vector Field — 231
 - 10.1.2 The Case of a Covector-Valued Form — 234
 - 10.2 The Reynolds Transport Theorem — 239
 - 10.3 Balance Equations in Integral Form — 240
 - 10.3.1 Power Balance — 240
 - 10.3.2 Mass Conservation — 242
 - 10.3.3 Transformation of the Spatial Power Balance Equation — 243
 - 10.4 Derivation of Differential Balance Equations Using the Covariance Principle — 244
 - 10.4.1 The Principle of Covariance — 244
 - 10.4.2 Change of Frame and Objective Transformations — 245
 - 10.4.3 Derivation of Spatial Conservation Laws — 246

- 11 The Evolutionary Problem – Examples — 249**
 - 11.1 Example: The Cylindrical Problem — 249
 - 11.1.1 Hollow Cylinders with Discrete Inhomogeneity — 249
 - 11.1.2 Hollow Cylinders with Continuous Inhomogeneity — 255
 - 11.1.3 Results and Discussion — 263
 - 11.2 Uniform Inflation of a Spherical Multilayered Structure — 267
 - 11.2.1 Layers and Assemblies — 267
 - 11.2.2 Coordinates and Vector Bases — 268
 - 11.2.3 Strain Measures — 271
 - 11.2.4 Incompressible Material — 271
 - 11.2.5 Compressible Material — 273
 - 11.2.6 Continuous Non-Euclidean Structures — 277
 - 11.2.7 Small Perturbations of the Self-Stressed Shape — 280
 - 11.3 Bending of Rectangular Blocks — 281
 - 11.3.1 Deformations of a Single Block — 281
 - 11.3.2 Stresses — 286
 - 11.3.3 Forces on Boundary Surfaces — 287
 - 11.3.4 Thin Layers — 289
 - 11.3.5 Discrete Accretion — 290
 - 11.3.6 Continuous Accretion — 292

12	Algebraic Structures — 297
12.1	Preliminary Comments on the Use of Sets — 297
12.2	Ordered Pairs. Cartesian Products. Relations — 297
12.3	Functions — 298
12.4	Some Algebraic Structures — 300
12.4.1	Groups — 300
12.4.2	Ring — 300
12.4.3	Module — 301
12.5	Linear Spaces and Mappings — 301
12.5.1	Vector Space over \mathbb{R} — 301
12.5.2	Linear and k -Linear Mappings — 303
12.5.3	Tensor Products of Vector Spaces — 305
12.5.4	Vectors and Linear Mappings in Euclidean Space — 308
12.6	Linear Groups — 311
12.7	Affine Space — 312
13	Review of Smooth Manifolds and Vector Bundles — 315
13.1	Smooth Manifolds — 315
13.1.1	Topological Spaces — 315
13.1.2	Smooth Structure — 321
13.1.3	Smooth Mappings — 322
13.1.4	Embedded Submanifolds — 325
13.2	The “Tower” of Tensor Spaces — 326
13.2.1	Tangent Space to a Smooth Manifold — 326
13.2.2	The tangent Map at a Point — 332
13.2.3	Cotangent Space to a Smooth Manifold — 333
13.2.4	Remarks on the Tangent Spaces — 334
13.2.5	The “Tower” — 336
13.2.6	Exterior Forms — 337
13.3	Vector Bundles and Their Sections — 340
13.3.1	Smooth Vector Bundles of Rank k — 340
13.3.2	Tangent and Cotangent Bundles — 341
13.3.3	Operations on Vector Bundles — 343
13.3.4	Vector Bundles of Higher Rank — 344
13.3.5	Sections of Vector Bundles — 345
13.3.6	Vector Bundle Homomorphisms — 347
13.3.7	Pullback and Pushforward — 348
13.3.8	Smooth Frames — 350
13.3.9	Exterior Differentiation — 351
13.3.10	The Riemannian Metric and Musical Isomorphisms — 352
13.4	Orientation and Integration on Manifolds — 354
13.4.1	The Volume Form and Orientation of Smooth Manifolds — 354

13.4.2	The Hodge Star Operator —	355
13.4.3	Integration of Differential Forms and Stokes' Theorem —	356
14	Connections on Principal Bundles —	359
14.1	Lie Groups and Lie Algebras —	359
14.1.1	Lie Groups and Homomorphisms —	359
14.1.2	Group Action —	359
14.1.3	Lie Algebra of the Lie Group —	361
14.1.4	Adjoint Representation —	363
14.1.5	Exponential Mapping —	363
14.2	Principal Bundles —	364
14.2.1	Bundles —	364
14.2.2	Principal Bundles —	365
14.2.3	Frame Bundles —	366
14.2.4	Associated Bundles —	367
14.3	Connections —	368
14.3.1	Connections on the Principal Bundle —	368
14.3.2	Local Representation of Connections —	369
14.3.3	Local Representation on Frame Bundles —	369
14.3.4	Gauge Maps —	370
14.3.5	Parallel Transport —	371
14.3.6	Curvature —	374
14.3.7	Torsion —	375
14.3.8	Bianchi Identities —	376
14.3.9	Covariant Derivatives on Associated Vector Bundles —	376
14.3.10	Direct Construction of Covariant Derivatives on Principal Bundles —	377

Bibliography — 381

Index — 387