

# Contents

	<i>Preface</i>	<i>page xi</i>
	Introduction	1
1	The concept of a manifold	4
	1.1 Topology and continuous maps	4
	1.2 Classes of smoothness of maps of Cartesian spaces	6
	1.3 Smooth structure, smooth manifold	7
	1.4 Smooth maps of manifolds	11
	1.5 A technical description of smooth surfaces in $\mathbb{R}^n$	16
	Summary of Chapter 1	20
2	Vector and tensor fields	21
	2.1 Curves and functions on $M$	22
	2.2 Tangent space, vectors and vector fields	23
	2.3 Integral curves of a vector field	30
	2.4 Linear algebra of tensors (multilinear algebra)	34
	2.5 Tensor fields on $M$	45
	2.6 Metric tensor on a manifold	48
	Summary of Chapter 2	53
3	Mappings of tensors induced by mappings of manifolds	54
	3.1 Mappings of tensors and tensor fields	54
	3.2 Induced metric tensor	60
	Summary of Chapter 3	63
4	Lie derivative	65
	4.1 Local flow of a vector field	65
	4.2 Lie transport and Lie derivative	70
	4.3 Properties of the Lie derivative	72
	4.4 Exponent of the Lie derivative	75
	4.5 Geometrical interpretation of the commutator $[V, W]$ , non-holonomic frames	77
	4.6 Isometries and conformal transformations, Killing equations	81
	Summary of Chapter 4	91

5	Exterior algebra	93
	5.1 Motivation: volumes of parallelepipeds	93
	5.2 $p$ -forms and exterior product	95
	5.3 Exterior algebra $\Lambda L^*$	102
	5.4 Interior product $i_v$	105
	5.5 Orientation in $L$	106
	5.6 Determinant and generalized Kronecker symbols	107
	5.7 The metric volume form	112
	5.8 Hodge (duality) operator $*$	118
	Summary of Chapter 5	125
6	Differential calculus of forms	126
	6.1 Forms on a manifold	126
	6.2 Exterior derivative	128
	6.3 Orientability, Hodge operator and volume form on $M$	133
	6.4 $V$ -valued forms	139
	Summary of Chapter 6	143
7	Integral calculus of forms	144
	7.1 Quantities under the integral sign regarded as differential forms	144
	7.2 Euclidean simplices and chains	146
	7.3 Simplices and chains on a manifold	149
	7.4 Integral of a form over a chain on a manifold	150
	7.5 Stokes' theorem	151
	7.6 Integral over a domain on an orientable manifold	153
	7.7 Integral over a domain on an orientable Riemannian manifold	159
	7.8 Integral and maps of manifolds	161
	Summary of Chapter 7	163
8	Particular cases and applications of Stokes' theorem	164
	8.1 Elementary situations	164
	8.2 Divergence of a vector field and Gauss' theorem	166
	8.3 Codifferential and Laplace–deRham operator	171
	8.4 Green identities	177
	8.5 Vector analysis in $E^3$	178
	8.6 Functions of complex variables	185
	Summary of Chapter 8	188
9	Poincaré lemma and cohomologies	190
	9.1 Simple examples of closed non-exact forms	191
	9.2 Construction of a potential on contractible manifolds	192
	9.3* Cohomologies and deRham complex	198
	Summary of Chapter 9	203
10	Lie groups: basic facts	204
	10.1 Automorphisms of various structures and groups	204

10.2	Lie groups: basic concepts	210
	Summary of Chapter 10	213
11	Differential geometry on Lie groups	214
11.1	Left-invariant tensor fields on a Lie group	214
11.2	Lie algebra $\mathcal{G}$ of a group $G$	222
11.3	One-parameter subgroups	225
11.4	Exponential map	227
11.5	Derived homomorphism of Lie algebras	230
11.6	Invariant integral on $G$	231
11.7	Matrix Lie groups: enjoy simplifications	232
	Summary of Chapter 11	243
12	Representations of Lie groups and Lie algebras	244
12.1	Basic concepts	244
12.2	Irreducible and equivalent representations, Schur's lemma	252
12.3	Adjoint representation, Killing–Cartan metric	259
12.4	Basic constructions with groups, Lie algebras and their representations	269
12.5	Invariant tensors and intertwining operators	278
12.6*	Lie algebra cohomologies	282
	Summary of Chapter 12	287
13	Actions of Lie groups and Lie algebras on manifolds	289
13.1	Action of a group, orbit and stabilizer	289
13.2	The structure of homogeneous spaces, $G/H$	294
13.3	Covering homomorphism, coverings $SU(2) \rightarrow SO(3)$ and $SL(2, \mathbb{C}) \rightarrow L_+^{\uparrow}$	299
13.4	Representations of $G$ and $\mathcal{G}$ in the space of functions on a $G$ -space, fundamental fields	310
13.5	Representations of $G$ and $\mathcal{G}$ in the space of tensor fields of type $\hat{\rho}$	319
	Summary of Chapter 13	325
14	Hamiltonian mechanics and symplectic manifolds	327
14.1	Poisson and symplectic structure on a manifold	327
14.2	Darboux theorem, canonical transformations and symplectomorphisms	336
14.3	Poincaré–Cartan integral invariants and Liouville's theorem	341
14.4	Symmetries and conservation laws	346
14.5*	Moment map	349
14.6*	Orbits of the coadjoint action	354
14.7*	Symplectic reduction	360
	Summary of Chapter 14	368
15	Parallel transport and linear connection on $M$	369
15.1	Acceleration and parallel transport	369
15.2	Parallel transport and covariant derivative	372
15.3	Compatibility with metric, RLC connection	382
15.4	Geodesics	389

15.5	The curvature tensor	401
15.6	Connection forms and Cartan structure equations	406
15.7	Geodesic deviation equation (Jacobi's equation)	418
15.8*	Torsion, complete parallelism and flat connection	422
	Summary of Chapter 15	428
16	Field theory and the language of forms	429
16.1	Differential forms in the Minkowski space $E^{1,3}$	430
16.2	Maxwell's equations in terms of differential forms	436
16.3	Gauge transformations, action integral	441
16.4	Energy–momentum tensor, space-time symmetries and conservation laws due to them	448
16.5*	Einstein gravitational field equations, Hilbert and Cartan action	458
16.6*	Non-linear sigma models and harmonic maps	467
	Summary of Chapter 16	476
17	Differential geometry on $TM$ and $T^*M$	478
17.1	Tangent bundle $TM$ and cotangent bundle $T^*M$	478
17.2	Concept of a fiber bundle	482
17.3	The maps $Tf$ and $T^*f$	485
17.4	Vertical subspace, vertical vectors	487
17.5	Lifts on $TM$ and $T^*M$	488
17.6	Canonical tensor fields on $TM$ and $T^*M$	494
17.7	Identities between the tensor fields introduced here	497
	Summary of Chapter 17	497
18	Hamiltonian and Lagrangian equations	499
18.1	Second-order differential equation fields	499
18.2	Euler–Lagrange field	500
18.3	Connection between Lagrangian and Hamiltonian mechanics, Legendre map	505
18.4	Symmetries lifted from the base manifold (configuration space)	508
18.5	Time-dependent Hamiltonian, action integral	518
	Summary of Chapter 18	522
19	Linear connection and the frame bundle	524
19.1	Frame bundle $\pi : LM \rightarrow M$	524
19.2	Connection form on $LM$	527
19.3	$k$ -dimensional distribution $\mathcal{D}$ on a manifold $\mathcal{M}$	530
19.4	Geometrical interpretation of a connection form: horizontal distribution on $LM$	538
19.5	Horizontal distribution on $LM$ and parallel transport on $M$	543
19.6	Tensors on $M$ in the language of $LM$ and their parallel transport	545
	Summary of Chapter 19	550
20	Connection on a principal $G$ -bundle	551
20.1	Principal $G$ -bundles	551

20.2	Connection form $\omega \in \Omega^1(P, \text{Ad})$	559
20.3	Parallel transport and the exterior covariant derivative $D$	563
20.4	Curvature form $\Omega \in \Omega^2(P, \text{Ad})$ and explicit expressions of $D$	567
20.5*	Restriction of the structure group and connection	576
	Summary of Chapter 20	585
21	Gauge theories and connections	587
21.1	Local gauge invariance: “conventional” approach	587
21.2	Change of section and a gauge transformation	594
21.3	Parallel transport equations for an object of type $\rho$ in a gauge $\sigma$	600
21.4	Bundle $P \times_{\rho} V$ associated to a principal bundle $\pi : P \rightarrow M$	606
21.5	Gauge invariant action and the equations of motion	607
21.6	Noether currents and Noether’s theorem	618
21.7*	Once more (for a while) on $LM$	626
	Summary of Chapter 21	633
22*	Spinor fields and the Dirac operator	635
22.1	Clifford algebras $C(p, q)$	637
22.2	Clifford groups $\text{Pin}(p, q)$ and $\text{Spin}(p, q)$	645
22.3	Spinors: linear algebra	650
22.4	Spin bundle $\pi : SM \rightarrow M$ and spinor fields on $M$	654
22.5	Dirac operator	662
	Summary of Chapter 22	670
Appendix A	Some relevant algebraic structures	673
A.1	Linear spaces	673
A.2	Associative algebras	676
A.3	Lie algebras	676
A.4	Modules	679
A.5	Grading	680
A.6	Categories and functors	681
Appendix B	Starring	683
	<i>Bibliography</i>	685
	<i>Index of (frequently used) symbols</i>	687
	<i>Index</i>	690