Contents

Preface vii

Introduction xv

Part I From groups to quantum groups 1

1 Hopf algebras 3
- 1.1 Motivation: Pontrjagin duality . . . 3
- 1.2 The concept of a Hopf algebra . . . 5
  - 1.2.1 Definition . . . 5
  - 1.2.2 Examples related to groups . . . 6
- 1.3 Axiomatics of Hopf algebras . . . 11
  - 1.3.1 Coalgebras and bialgebras . . . 12
  - 1.3.2 Convolution . . . 16
  - 1.3.3 Properties of the antipode . . . 18
  - 1.3.4 Another characterization of Hopf algebras . . . 22
  - 1.3.5 Hopf $*$-algebras . . . 26
- 1.4 The duality of Hopf algebras . . . 28
  - 1.4.1 The duality of finite-dimensional Hopf algebras . . . 28
  - 1.4.2 Dual pairings of Hopf algebras . . . 31
  - 1.4.3 The restricted dual of a Hopf algebra . . . 35

2 Multiplier Hopf algebras and their duality 40
- 2.1 Definition of multiplier Hopf algebras . . . 40
  - 2.1.1 Multipliers of algebras . . . 41
  - 2.1.2 Multiplier bialgebras . . . 42
  - 2.1.3 Multiplier Hopf algebras . . . 44
- 2.2 Integrals and their modular properties . . . 47
  - 2.2.1 The concept of an integral . . . 47
  - 2.2.2 Existence and uniqueness . . . 52
  - 2.2.3 The modular element of an integral . . . 54
  - 2.2.4 The modular automorphism of an integral . . . 56
- 2.3 Duality . . . 58
  - 2.3.1 The duality of regular multiplier Hopf algebras . . . 58
  - 2.3.2 The duality of algebraic quantum groups . . . 63

3 Algebraic compact quantum groups 65
3.1 Corepresentations of Hopf $*$-algebras 66
3.1.1 Definition and examples 66
3.1.2 Reformulation of the concept of a corepresentation 69
3.1.3 Construction of new corepresentations 75
3.2 Corepresentation theory and structure theory 80
3.2.1 Decomposition into irreducible corepresentations 80
3.2.2 Schur's orthogonality relations 83
3.2.3 Characterization of compact quantum groups 86
3.2.4 Characters of corepresentations 87
3.2.5 Modular properties of the Haar state 89
3.3 Discrete algebraic quantum groups 91

Part II Quantum groups and C^*-/von Neumann bialgebras 95

4 First definitions and examples 97
4.1 C^*-bialgebras and von Neumann bialgebras 97
4.2 Bialgebras associated to groups 100
4.3 Approaches to quantum groups in the setting of von Neumann algebras and C^*-algebras 104

5 C^*-algebraic compact quantum groups 107
5.1 Definition and examples 107
5.2 Corepresentations of C^*-bialgebras 111
5.2.1 Unitary corepresentations of C^*-algebraic compact quantum groups 111
5.2.2 Corepresentation operators of C^*-bialgebras 113
5.2.3 Constructions related to corepresentation operators 116
5.3 Corepresentation theory and structure theory 121
5.3.1 Decomposition into irreducible corepresentations 122
5.3.2 Schur's orthogonality relations 125
5.3.3 Characterization of C^*-algebraic compact quantum groups 127
5.4 The relation to algebraic compact quantum groups 128
5.4.1 From C^*-algebraic to algebraic CQGs 128
5.4.2 From algebraic to C^*-algebraic CQGs 130

6 Examples of compact quantum groups 135
6.1 Compact matrix quantum groups 136
6.2 The compact quantum group $\mathrm{SU}_\mu(2)$ 142
6.2.1 Definition and first properties 142
6.2.2 Corepresentations and their weights 145
6.2.3 Corepresentations and differential calculi 149

6.2.4 Modular properties of the Haar state 152
6.3 Products of compact quantum groups 154
6.4 The free unitary and the free orthogonal quantum groups 159

7 Multiplicative unitaries 166
7.1 The concept of a multiplicative unitary 167
7.1.1 Motivation 167
7.1.2 Definition and examples 168
7.2 The legs of a multiplicative unitary 172
7.2.1 Definition and first properties 172
7.2.2 Well-behaved multiplicative unitaries 175
7.2.3 Examples 179
7.2.4 The dual pairing, counit, and antipode of the legs 184
7.3 Classes of well-behaved multiplicative unitaries 189
7.3.1 Regular multiplicative unitaries 189
7.3.2 Manageable and modular multiplicative unitaries 197

8 Locally compact quantum groups 203
8.1 The concept of a locally compact quantum group 203
8.1.1 Weights 204
8.1.2 Locally compact quantum groups in the setting of von Neumann algebras 205
8.1.3 The modular automorphism group of a weight 207
8.1.4 Reduced C^*-algebraic quantum groups 210
8.2 Additional prerequisites 213
8.3 Main properties 216
8.3.1 The multiplicative unitary 217
8.3.2 The antipode and modular properties 218
8.3.3 The duality of locally compact quantum groups 222
8.3.4 Passage between the different levels 225
8.4 Examples of locally compact quantum groups 227
8.4.1 C^*-algebras generated by unbounded elements 229
8.4.2 The quantum groups $E_\mu(2)$ and $\widehat{E}_\mu(2)$ 233
8.4.3 The quantum $az + b$ group 242

Part III Selected topics 249

9 Coactions on C^*-algebras, reduced crossed products, and duality 251
9.1 Actions of groups and Takesaki–Takai duality 252
9.2 Coactions of C^*-bialgebras on C^*-algebras 256
9.3 Weak Kac systems 263

9.3.1 Balanced multiplicative unitaries 263
9.3.2 Weak Kac systems 267
9.3.3 Examples of weak Kac systems 268
9.4 Reduced crossed products and dual coactions 274
9.4.1 The reduced crossed product of a coaction of $A(V)$ 274
9.4.2 The dual coaction of a coaction of $A(V)$ 277
9.4.3 The dual coaction of a coaction of $\widehat{A}(V)$ 279
9.4.4 Comparison with the reduced crossed product of an action 280
9.5 Kac systems and the Baaj–Skandalis duality theorem 282
9.5.1 Kac systems 282
9.5.2 The Baaj–Skandalis duality theorem 286

10 Pseudo-multiplicative unitaries on Hilbert spaces 289
10.1 The relative tensor product of Hilbert modules 291
10.1.1 Hilbert modules over von Neumann algebras 291
10.1.2 Outline of the construction 293
10.1.3 Bounded elements of a Hilbert module 296
10.1.4 Construction of the relative tensor product 300
10.1.5 Properties of the relative tensor product 302
10.2 Hopf–von Neumann bimodules 307
10.2.1 The fiber product of von Neumann algebras 307
10.2.2 Hopf–von Neumann bimodules 312
10.3 Pseudo-multiplicative unitaries on Hilbert spaces 314
10.3.1 Definition 314
10.3.2 The legs of a pseudo-multiplicative unitary 316
10.3.3 The pseudo-multiplicative unitary of a groupoid 323

11 Pseudo-multiplicative unitaries on C^*-modules 328
11.1 Pseudo-multiplicative unitaries on C^*-modules 329
11.1.1 The flipped internal tensor product of C^*-modules 329
11.1.2 Definition and examples 330
11.1.3 Obstructions to the construction of the legs 335
11.2 Semigroup grading techniques on right C^*-bimodules 337
11.2.1 Homogeneous operators and C^*-families 337
11.2.2 Homogeneous elements of right C^*-bimodules 342
11.2.3 Examples related to groupoids 347
11.3 Hopf C^*-families 350
11.3.1 The internal tensor product of C^*-families 350
11.3.2 Morphisms of C^*-families 354
11.3.3 Hopf C^*-families 358
11.4 The legs of a decomposable pseudo-multiplicative unitary 359
11.5 Coactions of Hopf C^*-families 365

12 Appendix 369
12.1 C^*-algebras 369
12.2 C^*-modules 373
12.3 Von Neumann algebras 375
12.4 Slice maps 377
12.5 Auxiliary results 381

Bibliography 385

Symbol Index 397

Index 401