## Contents

	Notai	tion	page ix
	Forev	Foreword by Walter Hayman	
	Prefa	Preface	
	Intro	duction	1
1	Rear	Rearrangements	
	1.1	The Distribution Function	16
	1.2	The Decreasing Rearrangement	20
	1.3	Induced Measures	25
	1.4	Measure Preserving Transformations	31
	1.5	Nonatomic Measure Spaces	32
	1.6	Symmetric Decreasing Rearrangement on $\mathbb{R}^n$	39
	1.7	Polarization on $\mathbb{R}^n$	42
	1.8	Convergence Theorems for Rearrangements	48
	1.9	Notes and Comments	53
2	Main Inequalities on $\mathbb{R}^n$		
	2.1	Convex and AL Functions	55
	2.2	Main Inequalities for Two-Point Symmetrization	57
	2.3	Main Inequalities for Polarization	59
	2.4	Symmetrization Decreases the Modulus of Continuity	64
	2.5	Symmetrization Increases Certain Integrals in $\mathbb{R}^n$	68
	2.6	Proofs of the Uniqueness Statements	73
	2.7	Direct Consequences of the Main Inequalities	77
	2.8	Decomposition of Monotone and AL <sub>0</sub> Functions	82
	2.9	Proof of Theorem 2.15 for Discontinuous $\Psi$	88
	2.10	Notes and Comments	90

Contents

3	Diric	chlet Integral Inequalities	92
	3.1	Lipschitz Functions	92
	3.2	Symmetrization Decreases the p-Dirichlet Integral	
		of Lipschitz Functions	96
	3.3	Symmetrization Decreases the $\Phi$ -Dirichlet Integral of	
		Lipschitz Functions	101
	3.4	Sobolev Spaces $W^{1,p}(\mathbb{R}^n)$	106
	3.5	Weak Compactness	109
	3.6	Symmetrization Decreases the p-Dirichlet	
		Integral in $W^{1,p}(\mathbb{R}^n)$	112
	3.7	Continuity and Discontinuity of the Symmetric	
		Decreasing Rearrangement Operator	116
	3.8	Notes and Comments	117
4	Geor	netric Isoperimetric and Sharp Sobolev Inequalities	119
	4.1	Hausdorff Measures, Area Formula, and the	
		Gauss-Green Theorem	120
	4.2	Functions of Bounded Variation in $\mathbb{R}^n$	125
	4.3	Isoperimetric Inequalities for Perimeter and	
		Hausdorff Measure	128
	4.4	Isoperimetric Inequalities for Minkowski Content	132
	4.5	Coarea Formula	135
	4.6	Sharp Sobolev Embedding Constant for $p = 1$	139
	4.7	Sharp Sobolev Embedding Constants for $1$	143
	4.8	More about Sobolev Spaces	148
	4.9	Notes and Comments	151
5	Isoperimetric Inequalities for Physical Quantities		153
	5.1	Weak Solutions of $\Delta u = -f$	153
	5.2	Eigenvalues of the Laplacian	157
	5.3	Symmetrization Decreases the Principal Eigenvalue	159
	5.4	Domain Approximation Lemmas	163
	5.5	Symmetrization Decreases Newtonian Capacity	165
	5.6	Other Types of Capacity	171
	5.7	Symmetrization Increases Torsional Rigidity	
		and Mean Lifetime	178
	5.8	Notes and Comments	181
6	Steiner Symmetrization		
	6.1	Definition of Steiner Symmetrization	182
	6.2	Steiner Counterparts for Results in Chapter 1	185

vi

		Contents	vii
	6.3	Steiner Analogues for Two Simple Polarization Results	189
	6.4	Certain Integral Functionals Increase or Decrease	
		under Steiner Symmetrization	190
	6.5	Steiner Symmetrization Decreases the	
		Modulus of Continuity	192
	6.6	Steiner Symmetrization Decreases Dirichlet Integrals	195
	6.7	Proof of Lemma 6.18	204
	6.8	Steiner Symmetrization Decreases p-Dirichlet	
		Integrals in $W^{1,p}(\mathbb{R}^n)$	205
	6.9	Steiner Symmetrization Decreases Surface Area	210
	6.10	Steiner Symmetrization Increases or Decreases	
		Physical Quantities	213
	6.11	Notes and Comments	214
7	Sym	netrization on Spheres, and Hyperbolic	
	and (	Gauss Spaces	216
	7.1	The Sphere $\mathbb{S}^n$	216
	7.2	Spherical Coordinates on $\mathbb{S}^n$	219
	7.3	Inequalities for Spherical Symmetrization, Part 1	223
	7.4	Inequalities for Spherical Symmetrization, Part 2	229
	7.5	Cap Symmetrizations	234
	7.6	Hyperbolic Symmetrization	240
	7.7	Gauss Space Symmetrization	245
	7.8	Hölder Continuity of Quasiconformal Mappings	247
	7.9	Notes and Comments	253
8	Convolution and Beyond		254
	8.1	A Riesz-Type Convolution Inequality on $\mathbb{S}^1$	255
	8.2	Riesz's Convolution Inequality on $\mathbb R$	260
	8.3	The Riesz–Sobolev Inequality	261
	8.4	The Brunn–Minkowski Inequality	264
	8.5	The Brascamp-Lieb-Luttinger Inequality	267
	8.6	Symmetrization Increases the Trace of the Heat Kernel	270
	8.7	The Sharp Hardy–Littlewood–Sobolev Inequality	277
	8.8	Logarithmic Sobolev Inequalities	286
	8.9	Hypercontractivity	291
	8.10	Sharp Inequalities for Exponential Integrals	294
	8.11	Notes and Comments	297
9	The 🗲	t-Function	299
	9.1	The ★-Function on General Measure Spaces	299
	9.2	Preliminaries, and What Happens Next	300

	9.3	A Measurability Lemma	304
	9.4	Formulas for the Laplacian	305
	9.5	Pre-Subharmonicity on Shells	309
	9.6	The *-Function on Shells	315
	9.7	The $\star$ -Function on the Sphere	323
	9.8	The <b>*</b> -Function for Cap Symmetrization on	
		Ring-Type Domains	327
	9.9	Pre-Subharmonicity Theorem for s.d.r.	
		on Euclidean Domains	331
	9.10	The <b>*</b> -Function for s.d.r. on Euclidean Domains	342
	9.11	The *-Function for Steiner Symmetrization	
		on Euclidean Domains	346
	9.12	Notes and Comments	354
10	Comparison Principles for Semilinear Poisson PDEs		356
	10.1	Majorization	357
	10.2	Weakly Convex and Weakly Subharmonic Functions	362
	10.3	Comparison Principles for s.d.r. on Euclidean Domains	365
	10.4	Comparison Principle for Steiner Symmetrization	
		on Euclidean Domains	377
	10.5	Comparison Principle on the Sphere	381
	10.6	Comparison Principles on Shells	385
	10.7	Notes and Comments	395
11	The *-Function in Complex Analysis		398
	11.1	Introduction and Background	398
	11.2	The Nevanlinna Characteristic T and Its Extension $T^{\star}$	400
	11.3	Pólya Peaks and the Local Indicator of $T^*$	407
	11.4	Applications of $T^{\star}$ to Nevanlinna Theory	413
	11.5	Interlude: Subordination and Lehto's Theorem	427
	11.6	The <b>*</b> -Function and Univalent Functions	429
	11.7	Complements to the Univalent Integral Means Theorem	433
	11.8	Conjugate Functions	437
	11.9	Symmetrization and the Hyperbolic Metric	447
	11.10	Notes and Comments	452
	Refere	ences	454
	Index		469