

CONTENTS

I PROBABILITY IN CLASSICAL AND QUANTUM PHYSICS

1	Classical probability theory and stochastic processes	3
1.1	The probability space	3
1.1.1	The σ -algebra of events	3
1.1.2	Probability measures and Kolmogorov axioms	4
1.1.3	Conditional probabilities and independence	5
1.2	Random variables	5
1.2.1	Definition of random variables	5
1.2.2	Transformation of random variables	8
1.2.3	Expectation values and characteristic function	9
1.3	Stochastic processes	11
1.3.1	Formal definition of a stochastic process	11
1.3.2	The hierarchy of joint probability distributions	12
1.4	Markov processes	13
1.4.1	The Chapman–Kolmogorov equation	14
1.4.2	Differential Chapman–Kolmogorov equation	17
1.4.3	Deterministic processes and Liouville equation	19
1.4.4	Jump processes and the master equation	20
1.4.5	Diffusion processes and Fokker–Planck equation	27
1.5	Piecewise deterministic processes	31
1.5.1	The Liouville master equation	32
1.5.2	Waiting time distribution and sample paths	33
1.5.3	Path integral representation of PDPs	36
1.5.4	Stochastic calculus for PDPs	38
1.6	Lévy processes	44
1.6.1	Translation invariant processes	44
1.6.2	The Lévy–Khintchine formula	46
1.6.3	Stable Lévy processes	50
	References	55
2	Quantum probability	57
2.1	The statistical interpretation of quantum mechanics	57
2.1.1	Self-adjoint operators and the spectral theorem	57
2.1.2	Observables and random variables	61
2.1.3	Pure states and statistical mixtures	63
2.1.4	Joint probabilities in quantum mechanics	68
2.2	Composite quantum systems	72
2.2.1	Tensor product	72

2.2.2	Schmidt decomposition and entanglement	75
2.3	Quantum entropies	76
2.3.1	Von Neumann entropy	76
2.3.2	Relative entropy	78
2.3.3	Linear entropy	80
2.4	The theory of quantum measurement	80
2.4.1	Ideal quantum measurements	81
2.4.2	Operations and effects	83
2.4.3	Representation theorem for quantum operations	85
2.4.4	Quantum measurement and entropy	89
2.4.5	Approximate measurements	90
2.4.6	Indirect quantum measurements	93
2.4.7	Quantum non-demolition measurements	99
	References	101

II DENSITY MATRIX THEORY

3	Quantum master equations	105
3.1	Closed and open quantum systems	106
3.1.1	The Liouville–von Neumann equation	106
3.1.2	Heisenberg and interaction picture	108
3.1.3	Dynamics of open systems	110
3.2	Quantum Markov processes	113
3.2.1	Quantum dynamical semigroups	113
3.2.2	The Markovian quantum master equation	115
3.2.3	The adjoint quantum master equation	120
3.2.4	Multi-time correlation functions	121
3.2.5	Irreversibility and entropy production	123
3.3	Microscopic derivations	125
3.3.1	Weak-coupling limit	126
3.3.2	Relaxation to equilibrium	132
3.3.3	Singular-coupling limit	133
3.3.4	Low-density limit	134
3.4	The quantum optical master equation	136
3.4.1	Matter in quantized radiation fields	136
3.4.2	Decay of a two-level system	141
3.4.3	Decay into a squeezed field vacuum	144
3.4.4	More general reservoirs	147
3.4.5	Resonance fluorescence	148
3.4.6	The damped harmonic oscillator	155
3.5	Non-selective, continuous measurements	160
3.5.1	The quantum Zeno effect	161
3.5.2	Density matrix equation	162
3.6	Quantum Brownian motion	166

3.6.1	The Caldeira–Leggett model	167
3.6.2	High-temperature master equation	168
3.6.3	The exact Heisenberg equations of motion	176
3.6.4	The influence functional	186
3.7	Non-linear quantum master equations	195
3.7.1	Quantum Boltzmann equation	195
3.7.2	Mean field master equations	197
3.7.3	Mean field laser equations	199
3.7.4	Non-linear Schrödinger equation	202
3.7.5	Super-radiance	204
	References	209
4	Decoherence	212
4.1	The decoherence function	213
4.2	An exactly solvable model	218
4.2.1	Time evolution of the total system	218
4.2.2	Decay of coherences and the decoherence factor	220
4.2.3	Coherent subspaces and system-size dependence	223
4.3	Markovian mechanisms of decoherence	225
4.3.1	The decoherence rate	225
4.3.2	Quantum Brownian motion	226
4.3.3	Internal degrees of freedom	227
4.3.4	Scattering of particles	230
4.4	The damped harmonic oscillator	234
4.4.1	Vacuum decoherence	234
4.4.2	Thermal noise	238
4.5	Electromagnetic field states	242
4.5.1	Atoms interacting with a cavity field mode	243
4.5.2	Schrödinger cat states	248
4.6	Caldeira–Leggett model	254
4.6.1	General decoherence formula	254
4.6.2	Ohmic environments	256
4.7	Decoherence and quantum measurement	261
4.7.1	Dynamical selection of a pointer basis	261
4.7.2	Dynamical model for a quantum measurement	267
	References	270
 III STOCHASTIC PROCESSES IN HILBERT SPACE		
5	Probability distributions on Hilbert space	275
5.1	The state vector as a random variable in Hilbert space	275
5.1.1	A new type of quantum mechanical ensemble	275
5.1.2	Stern–Gerlach experiment	280
5.2	Probability density functionals on Hilbert space	283
5.2.1	Probability measures on Hilbert space	283

5.2.2	Distributions on projective Hilbert space	286
5.2.3	Expectation values	289
5.3	Ensembles of mixtures	290
5.3.1	Probability density functionals on state space	291
5.3.2	Description of selective quantum measurements	292
	References	293
6	Stochastic dynamics in Hilbert space	295
6.1	Dynamical semigroups and PDPs in Hilbert space	296
6.1.1	Reduced system dynamics as a PDP	296
6.1.2	The Hilbert space path integral	303
6.1.3	Diffusion approximation	305
6.1.4	Multi-time correlation functions	307
6.2	Stochastic representation of continuous measurements	312
6.2.1	Stochastic time evolution of \mathcal{E}_P -ensembles	313
6.2.2	Short-time behaviour of the propagator	313
6.3	Direct photodetection	316
6.3.1	Derivation of the PDP	316
6.3.2	Path integral solution	322
6.4	Homodyne photodetection	326
6.4.1	Derivation of the PDP for homodyne detection	327
6.4.2	Stochastic Schrödinger equation	331
6.5	Heterodyne photodetection	333
6.5.1	Stochastic Schrödinger equation	333
6.5.2	Stochastic collapse models	336
6.6	Stochastic density matrix equations	339
6.7	Photodetection on a field mode	341
6.7.1	The photocounting formula	341
6.7.2	QND measurement of a field mode	345
	References	349
7	The stochastic simulation method	352
7.1	Numerical simulation algorithms for PDPs	353
7.1.1	Estimation of expectation values	353
7.1.2	Generation of realizations of the process	354
7.1.3	Determination of the waiting time	355
7.1.4	Selection of the jumps	357
7.2	Algorithms for stochastic Schrödinger equations	358
7.2.1	General remarks on convergence	359
7.2.2	The Euler scheme	360
7.2.3	The Heun scheme	361
7.2.4	The fourth-order Runge–Kutta scheme	361
7.2.5	A second-order weak scheme	362
7.3	Examples	363
7.3.1	The damped harmonic oscillator	363

7.3.2	The driven two-level system	366
7.4	A case study on numerical performance	371
7.4.1	Numerical efficiency and scaling laws	371
7.4.2	The damped driven Morse oscillator	373
	References	379
8	Applications to quantum optical systems	381
8.1	Continuous measurements in QED	382
8.1.1	Constructing the microscopic Hamiltonian	382
8.1.2	Determination of the QED operation	384
8.1.3	Stochastic dynamics of multipole radiation	387
8.1.4	Representation of incomplete measurements	389
8.2	Dark state resonances	391
8.2.1	Waiting time distribution and trapping state	392
8.2.2	Measurement schemes and stochastic evolution	394
8.3	Laser cooling and Lévy processes	399
8.3.1	Dynamics of the atomic wave function	401
8.3.2	Coherent population trapping	406
8.3.3	Waiting times and momentum distributions	411
8.4	Strong field interaction and the Floquet picture	418
8.4.1	Floquet theory	419
8.4.2	Stochastic dynamics in the Floquet picture	421
8.4.3	Spectral detection and the dressed atom	424
	References	427
IV NON-MARKOVIAN QUANTUM PROCESSES		
9	Projection operator techniques	431
9.1	The Nakajima–Zwanzig projection operator technique	432
9.1.1	Projection operators	432
9.1.2	The Nakajima–Zwanzig equation	433
9.2	The time-convolutionless projection operator method	435
9.2.1	The time-local master equation	436
9.2.2	Perturbation expansion of the TCL generator	437
9.2.3	The cumulant expansion	441
9.2.4	Perturbation expansion of the inhomogeneity	442
9.2.5	Error analysis	445
9.3	Stochastic unravelling in the doubled Hilbert space	446
	References	448
10	Non-Markovian dynamics in physical systems	450
10.1	Spontaneous decay of a two-level system	451
10.1.1	Exact master equation and TCL generator	451
10.1.2	Jaynes–Cummings model on resonance	456
10.1.3	Jaynes–Cummings model with detuning	461

10.1.4 Spontaneous decay into a photonic band gap	464
10.2 The damped harmonic oscillator	465
10.2.1 The model and frequency renormalization	465
10.2.2 Factorizing initial conditions	466
10.2.3 The stationary state	471
10.2.4 Non-factorizing initial conditions	475
10.2.5 Disregarding the inhomogeneity	479
10.3 The spin-boson system	480
10.3.1 Microscopic model	480
10.3.2 Relaxation of an initially factorizing state	481
10.3.3 Equilibrium correlation functions	485
10.3.4 Transition from coherent to incoherent motion	486
References	487
V RELATIVISTIC QUANTUM PROCESSES	
11 Measurements in relativistic quantum mechanics	491
11.1 The Schwinger–Tomonaga equation	492
11.1.1 States as functionals of spacelike hypersurfaces	492
11.1.2 Foliations of space-time	496
11.2 The measurement of local observables	497
11.2.1 The operation for a local measurement	498
11.2.2 Relativistic state reduction	500
11.2.3 Multivalued space-time amplitudes	504
11.2.4 The consistent hierarchy of joint probabilities	507
11.2.5 EPR correlations	511
11.2.6 Continuous measurements	512
11.3 Non-local measurements and causality	516
11.3.1 Entangled quantum probes	517
11.3.2 Non-local measurement by EPR probes	520
11.3.3 Quantum state verification	525
11.3.4 Non-local operations and the causality principle	528
11.3.5 Restrictions on the measurability of operators	534
11.3.6 QND verification of non-local states	539
11.3.7 Preparation of non-local states	543
11.3.8 Exchange measurements	544
11.4 Quantum teleportation	546
11.4.1 Coherent transfer of quantum states	546
11.4.2 Teleportation and Bell-state measurement	549
11.4.3 Experimental realization	551
References	555
12 Open quantum electrodynamics	557
12.1 Density matrix theory for QED	558
12.1.1 Field equations and correlation functions	558

12.1.2	The reduced density matrix	565
12.2	The influence functional of QED	566
12.2.1	Elimination of the radiation degrees of freedom	566
12.2.2	Vacuum-to-vacuum amplitude	572
12.2.3	Second-order equation of motion	574
12.3	Decoherence by emission of bremsstrahlung	577
12.3.1	Introducing the decoherence functional	577
12.3.2	Physical interpretation	582
12.3.3	Evaluation of the decoherence functional	585
12.3.4	Path integral approach	595
12.4	Decoherence of many-particle states	602
	References	605
Index		607