Contents

Preface ix

Notation and Conventions xi

Introduction: An Overview of Some Problems of Unlikely Intersections 1

1 Unlikely Intersections in Multiplicative Groups and the Zilber Conjecture 15
1.1 Torsion points on subvarieties of $\mathbb{G}_{\mathrm{m}}^n$ . . . 16
1.2 Higher multiplicative rank . . . 22
1.3 Remarks on Theorem 1.3 and its developments . . . 29
1.3.1 Fields other than $\overline{\mathbb{Q}}$ . . . 29
1.3.2 Weakened assumptions . . . 29
1.3.3 Unlikely intersections of positive dimension and height bounds . . . 31
1.3.4 Unlikely intersections of positive dimension and Zilber's conjecture . . . 33
1.3.5 Unlikely intersections and reducibility of lacunary polynomials (Schinzel's conjecture) . . . 35
1.3.6 Zhang's notion of dependence . . . 36
1.3.7 Abelian varieties (and other algebraic groups) . . . 36
1.3.8 Uniformity of bounds . . . 37
Notes to Chapter 1 . . . 39
Sparseness of multiplicatively dependent points . . . 39
Other unlikely intersections . . . 39
A generalization of Theorem 1.3 . . . 40
An application of the methods to zeros of linear recurrences . . . 40
Comments on the Methods . . . 41

2 An Arithmetical Analogue 43
2.1 Some unlikely intersections in number fields . . . 43
2.2 Some applications of Theorem 2.1 . . . 48
2.3 An analogue of Theorem 2.1 for function fields . . . 50
2.4 Some applications of Theorem 2.2 . . . 52
2.5 A proof of Theorem 2.2 . . . 54
Notes to Chapter 2 . . . 58
Simplifying the proof of Theorem 1.3 . . . 58
Rational points on curves over $\mathbb{F}_p$ . . . 58
Unlikely Intersections and Holomorphic GCD in Nevanlinna Theory . . . 60

3 Unlikely Intersections in Elliptic Surfaces and Problems of Masser **62**
3.1 A method for the Manin-Mumford conjecture 62
3.2 Masser's questions on elliptic pencils 66
3.3 A finiteness proof 70
3.4 Related problems, conjectures, and developments 77
3.4.1 Pink's and related conjectures 77
3.4.2 Extending Theorem 3.3 from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ 80
3.4.3 Effectivity 83
3.4.4 Extending Theorem 3.3 to arbitrary pairs of points on families of elliptic curves 84
3.4.5 Simple abelian surfaces and Pell's equations over function fields 85
3.4.6 Further extensions and analogues 87
3.4.7 Dynamical analogues 89
Notes to Chapter 3 92
Torsion values for a single point: other arguments 92
A variation on the Manin-Mumford conjecture 93
Comments on the Methods 94

4 About the André-Oort Conjecture **96**
4.1 Generalities about the André-Oort Conjecture 96
4.2 Modular curves and complex multiplication 99
4.3 The theorem of André 105
4.3.1 An effective variation 111
4.4 Pila's proof of André's theorem 112
4.5 Shimura varieties 118
Notes to Chapter 4 123
Remarks on Edixhoven's approach to André's theorem 123
Some unlikely intersections beyond André-Oort 124
Definability and o-minimal structures 125

Appendix A Distribution of Rational Points on Subanalytic Surfaces **128**
by Umberto Zannier

Appendix B Uniformity in Unlikely Intersections: An Example for Lines in Three Dimensions **136**
by David Masser

Appendix C Silverman's Bounded Height Theorem for Elliptic Curves: A Direct Proof **138**
by David Masser

Appendix D Lower Bounds for Degrees of Torsion Points: The Transcendence Approach **140**
by David Masser

Appendix E A Transcendence Measure for a Quotient of Periods **143**
by David Masser

Appendix F Counting Rational Points on Analytic Curves: A Transcendence Approach 145
by David Masser

Appendix G Mixed Problems: Another Approach 147
by David Masser

Bibliography 149

Index 159