

CONTENTS

| | |
|---|-------------|
| PREFACE | xiii |
| ACKNOWLEDGMENTS | xv |
| 1 REVIEW OF PROBABILITY THEORY | 1 |
| 1.1 Introduction | 1 |
| 1.2 Random Variables | 1 |
| 1.2.1 Distribution Functions | 2 |
| 1.2.2 Discrete Random Variables | 3 |
| 1.2.3 Continuous Random Variables | 3 |
| 1.2.4 Expectations | 4 |
| 1.2.5 Moments of Random Variables and the Variance | 4 |
| 1.3 Transform Methods | 5 |
| 1.3.1 The Characteristic Function | 5 |
| 1.3.2 Moment-Generating Property of the Characteristic Function | 6 |
| 1.3.3 The s -Transform | 6 |
| 1.3.4 Moment-Generating Property of the s -Transform | 7 |
| 1.3.5 The z -Transform | 7 |
| 1.3.6 Moment-Generating Property of the z -Transform | 8 |
| 1.4 Covariance and Correlation Coefficient | 9 |

| | | |
|----------|--|-----------|
| 1.5 | Sums of Independent Random Variables | 10 |
| 1.6 | Some Probability Distributions | 11 |
| 1.6.1 | The Bernoulli Distribution | 11 |
| 1.6.2 | The Binomial Distribution | 12 |
| 1.6.3 | The Geometric Distribution | 12 |
| 1.6.4 | The Poisson Distribution | 13 |
| 1.6.5 | The Exponential Distribution | 13 |
| 1.6.6 | Normal Distribution | 14 |
| 1.7 | Limit Theorems | 16 |
| 1.7.1 | Markov Inequality | 16 |
| 1.7.2 | Chebyshev Inequality | 17 |
| 1.7.3 | Laws of Large Numbers | 17 |
| 1.7.4 | The Central Limit Theorem | 18 |
| | Problems | 19 |
| 2 | OVERVIEW OF STOCHASTIC PROCESSES | 21 |
| 2.1 | Introduction | 21 |
| 2.2 | Classification of Stochastic Processes | 22 |
| 2.3 | Mean and Autocorrelation Function | 22 |
| 2.4 | Stationary Processes | 23 |
| 2.4.1 | Strict-Sense Stationary Processes | 23 |
| 2.4.2 | Wide-Sense Stationary Processes | 24 |
| 2.5 | Power Spectral Density | 24 |
| 2.6 | Counting Processes | 25 |
| 2.7 | Independent Increment Processes | 25 |
| 2.8 | Stationary Increment Process | 25 |
| 2.9 | Poisson Processes | 26 |
| 2.9.1 | Compound Poisson Process | 28 |
| 2.10 | Markov Processes | 29 |
| 2.10.1 | Discrete-Time Markov Chains | 30 |
| 2.10.2 | State Transition Probability Matrix | 31 |
| 2.10.3 | The k -Step State Transition Probability | 31 |
| 2.10.4 | State Transition Diagrams | 32 |
| 2.10.5 | Classification of States | 33 |
| 2.10.6 | Limiting-State Probabilities | 34 |
| 2.10.7 | Doubly Stochastic Matrix | 35 |
| 2.10.8 | Continuous-Time Markov Chains | 35 |
| 2.10.9 | Birth and Death Processes | 36 |

| | | |
|----------|--|-----------|
| 2.11 | Gaussian Processes | 38 |
| 2.12 | Martingales | 38 |
| | 2.12.1 Stopping Times | 40 |
| | Problems | 41 |
| 3 | ONE-DIMENSIONAL RANDOM WALK | 44 |
| 3.1 | Introduction | 44 |
| 3.2 | Occupancy Probability | 46 |
| 3.3 | Random Walk as a Markov Chain | 49 |
| 3.4 | Symmetric Random Walk as a Martingale | 49 |
| 3.5 | Random Walk with Barriers | 50 |
| 3.6 | Mean-Square Displacement | 50 |
| 3.7 | Gambler's Ruin | 52 |
| | 3.7.1 Ruin Probability | 52 |
| | 3.7.2 Alternative Derivation of Ruin Probability | 54 |
| | 3.7.3 Duration of a Game | 55 |
| 3.8 | Random Walk with Stay | 56 |
| 3.9 | First Return to the Origin | 57 |
| 3.10 | First Passage Times for Symmetric Random Walk | 59 |
| | 3.10.1 First Passage Time via the Generating Function | 59 |
| | 3.10.2 First Passage Time via the Reflection Principle | 61 |
| | 3.10.3 Hitting Time and the Reflection Principle | 64 |
| 3.11 | The Ballot Problem and the Reflection Principle | 65 |
| | 3.11.1 The Conditional Probability Method | 66 |
| 3.12 | Returns to the Origin and the Arc-Sine Law | 67 |
| 3.13 | Maximum of a Random Walk | 72 |
| 3.14 | Two Symmetric Random Walkers | 73 |
| 3.15 | Random Walk on a Graph | 73 |
| | 3.15.1 Proximity Measures | 75 |
| | 3.15.2 Directed Graphs | 75 |
| | 3.15.3 Random Walk on an Undirected Graph | 76 |
| | 3.15.4 Random Walk on a Weighted Graph | 80 |
| 3.16 | Random Walks and Electric Networks | 80 |
| | 3.16.1 Harmonic Functions | 82 |
| | 3.16.2 Effective Resistance and Escape Probability | 82 |
| 3.17 | Correlated Random Walk | 85 |
| 3.18 | Continuous-Time Random Walk | 90 |
| | 3.18.1 The Master Equation | 92 |

| | | |
|----------|---|------------|
| 3.19 | Reinforced Random Walk | 94 |
| 3.19.1 | Polya's Urn Model | 94 |
| 3.19.2 | ERRW and Polya's Urn | 96 |
| 3.19.3 | ERRW Revisited | 97 |
| 3.20 | Miscellaneous Random Walk Models | 98 |
| 3.20.1 | Geometric Random Walk | 98 |
| 3.20.2 | Gaussian Random Walk | 99 |
| 3.20.3 | Random Walk with Memory | 99 |
| 3.21 | Summary | 100 |
| | Problems | 100 |
| 4 | TWO-DIMENSIONAL RANDOM WALK | 103 |
| 4.1 | Introduction | 103 |
| 4.2 | The Pearson Random Walk | 105 |
| 4.2.1 | Mean-Square Displacement | 105 |
| 4.2.2 | Probability Distribution | 107 |
| 4.3 | The Symmetric 2D Random Walk | 110 |
| 4.3.1 | Stirling's Approximation of Symmetric Walk | 112 |
| 4.3.2 | Probability of Eventual Return for Symmetric Walk | 113 |
| 4.3.3 | Mean-Square Displacement | 114 |
| 4.3.4 | Two Independent Symmetric 2D Random Walkers | 114 |
| 4.4 | The Alternating Random Walk | 115 |
| 4.4.1 | Stirling's Approximation of Alternating Walk | 117 |
| 4.4.2 | Probability of Eventual Return for Alternating Walk | 117 |
| 4.5 | Self-Avoiding Random Walk | 117 |
| 4.6 | Nonreversing Random Walk | 121 |
| 4.7 | Extensions of the NRRW | 126 |
| 4.7.1 | The Noncontinuing Random Walk | 126 |
| 4.7.2 | The Nonreversing and Noncontinuing Random Walk | 127 |
| 4.8 | Summary | 128 |
| 5 | BROWNIAN MOTION | 129 |
| 5.1 | Introduction | 129 |
| 5.2 | Brownian Motion with Drift | 132 |

| | | |
|----------|---|------------|
| 5.3 | Brownian Motion as a Markov Process | 132 |
| 5.4 | Brownian Motion as a Martingale | 133 |
| 5.5 | First Passage Time of a Brownian Motion | 133 |
| 5.6 | Maximum of a Brownian Motion | 135 |
| 5.7 | First Passage Time in an Interval | 135 |
| 5.8 | The Brownian Bridge | 136 |
| 5.9 | Geometric Brownian Motion | 137 |
| 5.10 | The Langevin Equation | 137 |
| 5.11 | Summary | 141 |
| | Problems | 141 |
| 6 | INTRODUCTION TO STOCHASTIC CALCULUS | 143 |
| 6.1 | Introduction | 143 |
| 6.2 | The Ito Integral | 145 |
| 6.3 | The Stochastic Differential | 146 |
| 6.4 | The Ito's Formula | 147 |
| 6.5 | Stochastic Differential Equations | 147 |
| 6.6 | Solution of the Geometric Brownian Motion | 148 |
| 6.7 | The Ornstein–Uhlenbeck Process | 151 |
| | 6.7.1 Solution of the Ornstein–Uhlenbeck SDE | 152 |
| | 6.7.2 First Alternative Solution Method | 153 |
| | 6.7.3 Second Alternative Solution Method | 154 |
| 6.8 | Mean-Reverting Ornstein–Uhlenbeck Process | 155 |
| 6.9 | Summary | 157 |
| 7 | DIFFUSION PROCESSES | 158 |
| 7.1 | Introduction | 158 |
| 7.2 | Mathematical Preliminaries | 159 |
| 7.3 | Diffusion on One-Dimensional Random Walk | 160 |
| | 7.3.1 Alternative Derivation | 163 |
| 7.4 | Examples of Diffusion Processes | 164 |
| | 7.4.1 Brownian Motion | 164 |
| | 7.4.2 Brownian Motion with Drift | 167 |
| 7.5 | Correlated Random Walk and the Telegraph Equation | 167 |
| 7.6 | Diffusion at Finite Speed | 170 |
| 7.7 | Diffusion on Symmetric Two-Dimensional Lattice Random Walk | 171 |
| 7.8 | Diffusion Approximation of the Pearson Random Walk | 173 |
| 7.9 | Summary | 174 |

| | | |
|----------|--|------------|
| 8 | LEVY WALK | 175 |
| 8.1 | Introduction | 175 |
| 8.2 | Generalized Central Limit Theorem | 175 |
| 8.3 | Stable Distribution | 177 |
| 8.4 | Self-Similarity | 182 |
| 8.5 | Fractals | 183 |
| 8.6 | Levy Distribution | 185 |
| 8.7 | Levy Process | 186 |
| 8.8 | Infinite Divisibility | 186 |
| | 8.8.1 The Infinite Divisibility of the Poisson Process | 187 |
| | 8.8.2 Infinite Divisibility of the Compound Poisson Process | 187 |
| | 8.8.3 Infinite Divisibility of the Brownian Motion with Drift | 188 |
| 8.9 | Levy Flight | 188 |
| | 8.9.1 First Passage Time of Levy Flights | 190 |
| | 8.9.2 Leapover Properties of Levy Flights | 190 |
| 8.10 | Truncated Levy Flight | 191 |
| 8.11 | Levy Walk | 191 |
| | 8.11.1 Levy Walk as a Coupled CTRW | 192 |
| | 8.11.2 Truncated Levy Walk | 195 |
| 8.12 | Summary | 195 |
| 9 | FRACTIONAL CALCULUS AND ITS APPLICATIONS | 196 |
| 9.1 | Introduction | 196 |
| 9.2 | Gamma Function | 197 |
| 9.3 | Mittag–Leffler Functions | 198 |
| 9.4 | Laplace Transform | 200 |
| 9.5 | Fractional Derivatives | 202 |
| 9.6 | Fractional Integrals | 203 |
| 9.7 | Definitions of Fractional Integro-Differentials | 203 |
| | 9.7.1 Riemann–Liouville Fractional Derivative | 204 |
| | 9.7.2 Caputo Fractional Derivative | 205 |
| | 9.7.3 Grunwald–Letnikov Fractional Derivative | 206 |
| 9.8 | Fractional Differential Equations | 207 |
| | 9.8.1 Relaxation Differential Equation of Integer Order | 208 |
| | 9.8.2 Oscillation Differential Equation of Integer Order | 208 |
| | 9.8.3 Relaxation and Oscillation Fractional Differential Equations | 209 |

| | | |
|-----------|--|------------|
| 9.9 | Applications of Fractional Calculus | 210 |
| 9.9.1 | Fractional Brownian Motion | 210 |
| 9.9.2 | Multifractional Brownian Motion | 213 |
| 9.9.3 | Fractional Random Walk | 213 |
| 9.9.4 | Fractional (or Anomalous) Diffusion | 215 |
| 9.9.5 | Fractional Gaussian Noise | 221 |
| 9.9.6 | Fractional Poisson Process | 222 |
| 9.10 | Summary | 224 |
| 10 | PERCOLATION THEORY | 225 |
| 10.1 | Introduction | 225 |
| 10.2 | Graph Theory Revisited | 226 |
| 10.2.1 | Complete Graphs | 226 |
| 10.2.2 | Random Graphs | 226 |
| 10.3 | Percolation on a Lattice | 228 |
| 10.3.1 | Cluster Formation and Phase Transition | 229 |
| 10.3.2 | Percolation Probability and Critical Exponents | 233 |
| 10.4 | Continuum Percolation | 235 |
| 10.4.1 | The Boolean Model | 235 |
| 10.4.2 | The Random Connection Model | 236 |
| 10.5 | Bootstrap (or k -Core) Percolation | 237 |
| 10.6 | Diffusion Percolation | 237 |
| 10.6.1 | Bootstrap Percolation versus Diffusion Percolation | 239 |
| 10.7 | First-Passage Percolation | 239 |
| 10.8 | Explosive Percolation | 240 |
| 10.9 | Percolation in Complex Networks | 242 |
| 10.9.1 | Average Path Length | 243 |
| 10.9.2 | Clustering Coefficient | 243 |
| 10.9.3 | Degree Distribution | 244 |
| 10.9.4 | Percolation and Network Resilience | 244 |
| 10.10 | Summary | 245 |
| | REFERENCES | 247 |
| | INDEX | 253 |