

UNIVERSITY LECTURE SERIES VOLUME 72

Quantum Field Theory: Batalin–Vilkovisky Formalism and Its Applications

Pavel Mnev

 **AMS** AMERICAN
MATHEMATICAL
SOCIETY
Providence, Rhode Island

Contents

Preface	ix
Chapter 1. Introduction	1
1.1. Prologue	1
1.2. Atiyah-Segal picture of quantum field theory	5
1.2.1. Atiyah's axioms of topological quantum field theory	6
1.2.2. Segal's QFT	10
1.2.3. Example of a TQFT: Dijkgraaf-Witten model	12
1.3. The idea of path integral construction of quantum field theory	14
1.3.1. Classical field theory data	14
1.3.2. Idea of path integral quantization	14
1.3.3. Heuristic argument for gluing	15
1.3.4. How to define path integrals?	15
1.3.5. Towards Batalin-Vilkovisky (BV) formalism	17
1.4. Plan of the exposition	18
Chapter 2. Classical Chern-Simons theory	21
2.1. Chern-Simons theory on a closed 3-manifold	21
2.1.1. Fields	21
2.1.2. Action	21
2.1.3. Euler-Lagrange equation	21
2.1.4. Gauge symmetry	22
2.1.5. Chern-Simons invariant on the moduli space of flat connections	23
2.1.6. Remark: more general G	24
2.1.7. Relation to the second Chern class	24
2.2. Chern-Simons theory on manifolds with boundary	25
2.2.1. Phase space	25
2.2.2. δS_{CS} , Euler-Lagrange equations	26
2.2.3. Noether 1-form, symplectic structure on the phase space	26
2.2.4. "Cauchy subspace"	27
2.2.5. $L_{M,\Sigma}$	28
2.2.6. Reduction of the boundary structure by gauge transformations	29
2.2.7. Lagrangian property of $L_{M,\Sigma}$	30
2.2.8. Behavior of S_{CS} under gauge transformations, Wess-Zumino cocycle	31
2.2.9. Prequantum line bundle on the moduli space of flat connections on the surface	32
2.2.10. Two exciting formulae	33
2.2.11. Classical field theory as a functor to the symplectic category	34

Chapter 3. Feynman diagrams	37
3.1. Gauss and Fresnel integrals	37
3.2. Stationary phase formula	38
3.3. Gaussian expectation values. Wick's lemma	41
3.4. A reminder on graphs and graph automorphisms	44
3.5. Back to integrals: Gaussian expectation value of a product of homogeneous polynomials	47
3.6. Perturbed Gaussian integral	48
3.6.1. Aside: Borel summation	53
3.6.2. Connected graphs	54
3.6.3. Introducing the "Planck constant" and bookkeeping by Euler characteristic of Feynman graphs	55
3.6.4. Expectation values with respect to perturbed Gaussian measure	56
3.6.5. Fresnel (oscillatory) version of perturbative integral	57
3.6.6. Perturbation expansion via the exponential of a second order differential operator	57
3.7. Stationary phase formula with corrections	58
3.7.1. Laplace method	59
3.8. Berezin integral	61
3.8.1. Odd vector spaces	61
3.8.2. Integration on the odd line	61
3.8.3. Integration on the odd vector space	61
3.9. Gaussian integral over an odd vector space	63
3.10. Perturbative integral over a vector superspace	64
3.10.1. "Odd Wick's lemma"	64
3.10.2. Perturbative integral over an odd vector space	64
3.10.3. Perturbative integral over a superspace	66
3.11. Digression: the logic of perturbative path integral	68
3.11.1. Example: scalar theory with ϕ^3 interaction	69
3.11.2. Divergencies!	70
3.11.3. Regularization and renormalization	71
3.11.4. Wilson's picture of renormalization ("Wilson's RG flow")	72
Chapter 4. Batalin-Vilkovisky formalism	75
4.1. Faddeev-Popov construction	75
4.1.1. Hessian of S_{FP} in an adapted chart	78
4.1.2. Stationary phase evaluation of Faddeev-Popov integral	79
4.1.3. Motivating example: Yang-Mills theory	82
4.2. Elements of supergeometry	84
4.2.1. Supermanifolds	84
4.2.2. \mathbb{Z} -graded (super)manifolds	86
4.2.3. Differential graded manifolds (a.k.a. Q -manifolds)	87
4.2.4. Integration on supermanifolds	92
4.2.5. Change of variables formula for integration over supermanifolds	93
4.2.6. Divergence of a vector field	94
4.3. BRST formalism	95
4.3.1. Classical BRST formalism	95
4.3.2. Quantum BRST formalism	96
4.3.3. Faddeev-Popov via BRST	97

4.3.4.	Remark: reducible symmetries and higher ghosts	100
4.4.	Odd-symplectic manifolds	100
4.4.1.	Differential forms on super (graded) manifolds	100
4.4.2.	Odd-symplectic supermanifolds	101
4.4.3.	Odd-symplectic manifolds with a compatible Berezinian. BV Laplacian.	103
4.4.4.	BV integrals. Stokes' theorem for BV integrals.	104
4.5.	Algebraic picture: BV algebras. Master equation and canonical transformations of its solutions	106
4.5.1.	BV algebras	106
4.5.2.	Classical and quantum master equation	108
4.5.3.	Canonical transformations	109
4.6.	Half-densities on odd-symplectic manifolds. Canonical BV Laplacian. Integral forms	109
4.6.1.	Half-densities on odd-symplectic manifolds	109
4.6.2.	Canonical BV Laplacian on half-densities	111
4.6.3.	Integral forms	112
4.7.	Fiber BV integrals	113
4.8.	Batalin-Vilkovisky formalism	115
4.8.1.	Classical BV formalism	115
4.8.2.	Quantum BV formalism	116
4.8.3.	Faddeev-Popov via BV	118
4.8.4.	BV for gauge symmetry given by a non-integrable distribution	121
4.8.5.	Felder-Kazhdan existence-uniqueness result for solutions of the classical master equation	126
4.9.	AKSZ sigma models	129
4.9.1.	AKSZ construction	129
4.9.2.	Example: Chern-Simons theory	134
4.9.3.	Example: Poisson sigma model	137
4.9.4.	Example: BF theory	140
Chapter 5.	Applications	147
5.1.	Cellular BF theory	147
5.1.1.	Abstract BF theory associated to a dgLa	147
5.1.2.	Effective action induced on a subcomplex	149
5.1.3.	Geometric situation	154
5.1.4.	Remarks	161
5.2.	Perturbative Chern-Simons theory	162
5.2.1.	Perturbative contribution of an acyclic flat connection: one-loop part	162
5.2.2.	Higher loop corrections, after Axelrod-Singer	165
5.3.	Kontsevich's deformation quantization via Poisson sigma model	170
5.3.1.	Associativity: a heuristic path integral argument	173
5.3.2.	Associativity from Stokes' theorem on configuration spaces	174
5.3.3.	Kontsevich's L_∞ morphism	175
Bibliography		179
Index		185