Contents

Part I Spin Systems and Critical Phenomena

1 Spin Systems 3
1.1 Critical Phenomena and the Renormalisation Group 3
1.2 Ising Model 5
1.3 Spin Systems and Universality 8
1.4 Mean-Field Model 10
1.4.1 Critical Behaviour of the Mean-Field Model 10
1.4.2 Renormalised Measure 12
1.4.3 Magnetisation and Susceptibility: Proof of Theorem 1.4.1 13
1.5 Gaussian Free Field and Simple Random Walk 17
1.5.1 Gaussian Free Field 17
1.5.2 Simple Random Walk 19
1.5.3 The Bubble Diagram 21
1.6 $|\varphi|^4$ Model 23
1.6.1 Definition of the $|\varphi|^4$ Model 23
1.6.2 Critical Exponents of the $|\varphi|^4$ Model 25
1.7 Self-Avoiding Walk 28

2 Gaussian Fields 29
2.1 Gaussian Integration 29
2.2 Cumulants 35

3 Finite-Range Decomposition 37
3.1 Progressive Integration 37
3.2 Finite-Range Decomposition: Continuum 39
3.3 Finite-Range Decomposition: Lattice 42
3.3.1 Statement of the Decomposition 42
3.3.2 Integral Decomposition 43

3.3.3 Chebyshev Polynomials 46
3.3.4 Proof of Proposition 3.3.1 48
3.4 Finite-Range Decomposition: Torus 51

4 The Hierarchical Model 53
4.1 Hierarchical GFF 53
4.1.1 Hierarchical Fields 53
4.1.2 Construction of Hierarchical GFF 55
4.1.3 Properties of Hierarchical Covariances 58
4.2 Hierarchical $|\varphi|^4$ Model 61
4.3 GFF and $|\varphi|^4$ Model 62

Part II The Renormalisation Group: Perturbative Analysis

5 The Renormalisation Group Map 69
5.1 Reduction to Block Analysis 69
5.1.1 Progressive Integration 69
5.1.2 Polynomials in the Hierarchical Field 70
5.1.3 Functionals of the Hierarchical Field 72
5.1.4 Global to Local Reduction 72
5.2 The Renormalisation Group Map 74
5.2.1 Local Coordinates 74
5.2.2 Localisation 75
5.2.3 Perturbative Map 78
5.2.4 Definition of the Renormalisation Group Map 80
5.2.5 The Last Renormalisation Group Step 82
5.3 Perturbative Flow of Coupling Constants: The Map Φ_{pt} 83
5.3.1 Statement of the Perturbative Flow 83
5.3.2 Proof of the Perturbative Flow 85

6 Flow Equations and Main Result 89
6.1 Analysis of Perturbative Flow 89
6.1.1 Flow of $\bar{g}$ 90
6.1.2 Perturbative Stable Manifold 93
6.2 Reduction of Proof of Theorem 4.2.1 97

Part III The Renormalisation Group: Nonperturbative Analysis

7 The T_z-Seminorm 107
7.1 Definition of the T_z-Seminorm 107
7.2 Control of Derivatives 111
7.3 Expectation and the T_z-Seminorm 113
7.4 Exponentials and the T_z-Seminorm 114
7.5 Taylor's Theorem and the T_z-Seminorm 115
7.6 Polynomial Estimates 119

8 Global Flow: Proof of Theorem 4.2.1 123
8.1 Fluctuation and Block-Spin Fields 123
8.1.1 Hierarchical Field 123
8.1.2 Fluctuation Field 124
8.1.3 Block-Spin Field 125
8.2 Main Estimate on Renormalisation Group Map 126
8.2.1 Domain for V 126
8.2.2 Norms 126
8.2.3 Main Result 128
8.3 Construction of Critical Point 130
8.4 Proof of Theorem 6.2.1 134

9 Nonperturbative Contribution to Φ_+^U: Proof of Theorem 8.2.5 139
9.1 The Polynomial R_+^U 139
9.2 The Standard and Extended Norms 140
9.2.1 Utility of the Extended Norm 140
9.2.2 Choice of the Space $\mathcal{Y}$ 141
9.3 Norms of Polynomials 143
9.4 Proof of Theorem 9.1.1 148

10 Bounds on Φ_+^K: Proof of Theorem 8.2.4 151
10.1 Main Result 151
10.2 Stability 156
10.3 Bound on S_0: Proof of Lemma 10.1.4 160
10.3.1 Estimation of Gaussian Moments 160
10.3.2 Bound on S_0 161
10.4 Bound on S_1: Proof of Lemma 10.1.5 164
10.5 Crucial Contraction 168
10.6 Continuity in the Mass 173
10.7 Last Renormalisation Group Step: Proof of Proposition 6.2.2 ... 176

Part IV Self-Avoiding Walk and Supersymmetry

11 Self-Avoiding Walk and Supersymmetry 183
11.1 Critical Behaviour of Self-Avoiding Walk 184
11.1.1 Self-Avoiding Walk 184
11.1.2 Continuous-Time Random Walk 187
11.1.3 Continuous-Time Weakly Self-Avoiding Walk 188
11.2 Random Walk Representation of Spin Systems 190
11.2.1 Continuous-Time Random Walk and the Laplacian 190
11.2.2 BFS–Dynkin Isomorphism 192
11.3 Supersymmetric Representation 194
11.3.1 The Case $n = 0$ 194
11.3.2 Integration of Differential Forms 195
11.3.3 Localisation Theorem and Weakly Self-Avoiding Walk 200

11.3.4 Localisation Theorem and Strictly Self-Avoiding Walk 202
11.4 Supersymmetry and the Localisation Theorem 204
11.4.1 The Localisation Theorem 205
11.4.2 Supersymmetry and Exterior Calculus 209

Part V Appendices

12 Appendix A: Extension to Euclidean Models 213
12.1 Perturbative Renormalisation Group Coordinate 214
12.2 Approximate Factorisation 215
12.2.1 Factorisation of Expectation 216
12.2.2 Circle Product 216
12.3 Change of Coordinates 217
12.3.1 Block Cancellation 218
12.3.2 Small Set Cancellation 218
12.3.3 Application 219
12.4 Expectation, Change of Scale, and Reblocking 220
12.5 Cancellation via Change of Coordinates 223
12.5.1 Local Cancellation: Perturbative 223
12.5.2 Local Cancellation: Nonperturbative 226
12.6 Norms 229
12.6.1 T_φ-Seminorms 229
12.6.2 Regulators 231

13 Appendix B: Solutions to Exercises 235
13.1 Chapter 1 Exercises 235
13.2 Chapter 2 Exercises 240
13.3 Chapter 3 Exercises 244
13.4 Chapter 4 Exercises 247
13.5 Chapter 5 Exercises 253
13.6 Chapter 6 Exercises 256
13.7 Chapter 7 Exercises 261
13.8 Chapter 8 Exercises 262
13.9 Chapter 9 Exercises 264
13.10 Chapter 11 Exercises 264

References 271

Index 279