Contents

1 Topology of Singular Spaces: Motivation, Overview 1
1.1 Poincaré Duality 1
1.2 Topology of Projective Manifolds: Kähler Package 4
Hodge Decomposition 5
Lefschetz Hyperplane Section Theorem 6
Hard Lefschetz Theorem 8

2 Intersection Homology: Definition, Properties 11
2.1 Topological Pseudomanifolds 11
2.2 Borel–Moore Homology 13
2.3 Intersection Homology via Chains 16
2.4 Normalization 26
2.5 Intersection Homology of an Open Cone 29
2.6 Poincaré Duality for Pseudomanifolds 31
2.7 Signature of Pseudomanifolds 32

3 L-Classes of Stratified Spaces 37
3.1 Multiplicative Characteristic Classes of Vector Bundles: Examples 38
3.2 Characteristic Classes of Manifolds: Tangential Approach 40
3.3 L-Classes of Manifolds: Pontrjagin–Thom Construction 44
Construction 45
Coincidence with Hirzebruch L-Classes 47
Removing the Dimension Restriction 48
3.4 Goresky–MacPherson L-Classes 49

4 Brief Introduction to Sheaf Theory 53
4.1 Sheaves 53
4.2 Local Systems 58
Homology with Local Coefficients 60
Intersection Homology with Local Coefficients 62
4.3 Sheaf Cohomology 63

4.4 Complexes of Sheaves 66
4.5 Homotopy Category 71
4.6 Derived Category 73
4.7 Derived Functors 76

5 Poincaré–Verdier Duality 81
5.1 Direct Image with Proper Support 81
5.2 Inverse Image with Compact Support 85
5.3 Dualizing Functor 86
5.4 Verdier Dual via the Universal Coefficient Theorem 88
5.5 Poincaré and Alexander Duality on Manifolds 89
5.6 Attaching Triangles. Hypercohomology Long Exact Sequences of Pairs 91

6 Intersection Homology After Deligne 93
6.1 Introduction 93
6.2 Intersection Cohomology Complex 94
6.3 Deligne's Construction of Intersection Homology 98
6.4 Generalized Poincaré Duality 102
6.5 Topological Invariance of Intersection Homology 106
6.6 Rational Homology Manifolds 108
6.7 Intersection Homology Betti Numbers, I 111

7 Constructibility in Algebraic Geometry 117
7.1 Definition: Properties 117
7.2 Local Calculus 119
7.3 Euler Characteristics of Constructible Complexes. Applications ... 122

8 Perverse Sheaves 129
8.1 Definition, Examples 130
8.2 Gluing of t-Structures 135
8.3 Examples of Perverse Sheaves 137
8.4 Intermediate Extension 140
8.5 A Splitting Criterion for Perverse Sheaves 144
8.6 Artin's Vanishing Theorem 146

9 The Decomposition Package and Applications 149
9.1 Lefschetz Hyperplane Section Theorem 149
9.2 Hard Lefschetz Theorem for Intersection Cohomology 151
9.3 The BBDG Decomposition Theorem and Applications 153
Stratifications of Algebraic Maps 153
Deligne's Decomposition Theorem 154
Semi-Small Maps 156
The Decomposition Theorem for Semi-Small Maps 160
The BBDG Decomposition Theorem for Arbitrary Maps 162
First Applications of the Decomposition Theorem 164
Set of Supports of an Algebraic Map 169

9.4 Applications of the Decomposition Package 170
Topology of Hilbert Schemes of Points on a Smooth Complex Surface 171
Stanley's Proof of McMullen's Conjecture 174
Huh–Wang's Proof of Dowling–Wilson's Conjecture 177

10 Hypersurface Singularities. Nearby and Vanishing Cycles 181
10.1 Brief Overview of Complex Hypersurface Singularities 181
10.2 Global Aspects of Hypersurface Singularities 189
10.3 Nearby and Vanishing Cycles 196
Construction 197
Relation with Perverse Sheaves 202
Thom–Sebastiani for Vanishing Cycles 203
On Euler Characteristic Computations 205
10.4 Euler Characteristics of Complex Projective Hypersurfaces 207
10.5 Generalized Riemann–Hurwitz-Type Formulae 211
10.6 Homological Connectivity of Milnor Fiber and Link of Hypersurface Singularity Germs 214
10.7 Canonical and Variation Morphisms 216

11 Overview of Saito's Mixed Hodge Modules, and Immediate Applications 221
11.1 Classical Hodge Theory 221
11.2 Mixed Hodge Modules 227
11.3 Hodge Theory on Intersection Cohomology Groups 235
11.4 Intersection Homology Betti Numbers, II 241

12 Epilogue 245
12.1 Applications to Enumerative Geometry 245
12.2 Characteristic Classes of Complex Algebraic Varieties and Applications 247
12.3 Perverse Sheaves on Semi-Abelian Varieties: Cohomology Jump Loci, Propagation, Generic Vanishing 248
12.4 Generic Vanishing Theory via Mixed Hodge Modules 251
12.5 Applications to Representation Theory 253
12.6 Alexander-Type Invariants of Complex Hypersurface Complements 253

Bibliography 255

Index 267