Contents

1 Preliminary Material 1
1.1 Holomorphic Mappings 2
1.1.1 Definitions 2
1.1.2 The Case Where E is Finite-Dimensional 6
1.1.3 Holomorphic = Analytic for Arbitrary E 13
1.2 Complex Manifolds 16
1.2.1 Manifolds 16
1.2.2 Vector Bundles 20
1.2.3 Projective Space $\mathbb{P}(E)$ 26
1.2.4 Grassmannians 29
1.2.5 Further Examples of Complex Manifolds 36
1.2.6 Integration on Oriented Manifolds and Stokes' Formula 36
1.2.7 Differential Forms on a Complex Manifold 41
1.3 Symmetric Products of $\mathbb{C}$ 43
1.3.1 Continuity of Roots 43
1.3.2 Weierstrass Preparation Theorem 54
1.4 The Symmetric Product of $\mathbb{C}^p$ 63
1.4.1 Symmetric Products of Topological Spaces 64
1.4.2 Vector Symmetric Functions 67
1.4.3 Vertical Localization 74
1.4.4 Canonical Equations 77
1.4.5 Complex Structure on $\mathrm{Sym}^k(\mathbb{C}^p)$ 79
1.4.6 Stratification 88

2 Multigraphs and Reduced Complex Spaces 93
2.1 Reduced Multigraph 94
2.1.1 Proper Mappings 94
2.1.2 Analytic Subsets 101
2.1.3 Analytic Continuation 105
2.1.4 Analytic Étale Coverings 112

2.1.5 Reduced Multigraphs 114
2.1.6 Local Study of Reduced Multigraphs 120
2.1.7 Irreducibility of Reduced Multigraphs 123
2.2 Multigraphs 128
2.2.1 Basic Definitions 128
2.2.2 Classification Map and the Canonical Equation 130
2.3 Analytic Subsets 132
2.3.1 Local Parameterization Theorem: First Version 132
2.3.2 Irreducible Components and Singular Locus 138
2.3.3 Maximum Principle 147
2.3.4 Ramified Covers 149
2.3.5 Local Parameterization Theorem: Final Version 154
2.3.6 Analyticity of the Singular Locus 159
2.4 Reduced Complex Spaces 160
2.4.1 Definitions and Elementary Properties 160
2.4.2 Singular Locus and Irreducible Components 167
2.4.3 Dimension and Local Irreducibility 169
2.4.4 Minimal Embeddings and the Zariski Tangent Space 175
2.4.5 Fiber Dimension and Generic Rank 180
2.4.6 Symmetric Product of a Reduced Complex Space 187
2.4.7 Proper Finite Mappings 191
2.4.8 Remmert–Stein Theorem 194
2.5 Notes on Chapter 1 and this chapter 201
2.5.1 The Preparation and Division Theorems 201
2.5.2 The Local Parameterization Theorem 201
2.5.3 The Three Definitions of Ramified Covers 202
2.5.4 Complex Spaces 205
2.5.5 The Theorem of Remmert–Stein 206

3 Analysis and Geometry on a Reduced Complex Space 207
3.1 Transversality and the Zariski Tangent Cone 208
3.1.1 Transverse Planes to an Analytic Subset 208
3.1.2 Algebraic Cones 216
3.1.3 Transversality and the Tangent Cone 223
3.1.4 Zariski Tangent Cycle 231
3.2 The Theorem of P. Lelong 237
3.2.1 Introduction 237
3.2.2 Preliminaries 238
3.2.3 The Case of a Reduced Multigraph 242
3.2.4 Differential Forms on a Reduced Complex Space 250
3.2.5 Lelong's Theorem : General Case 262
3.2.6 Volume 264
3.3 Coherent Sheaves 265
3.3.1 Coherent Sheaves on a Reduced Complex Space 265
3.3.2 Canonical Topology 277

3.4 Modifications and Blowups 285
3.4.1 Modifications 286
3.4.2 Blowups 288
3.4.3 Meromorphic Mappings 293
3.5 Normalization 294
3.5.1 Normal Spaces 294
3.5.2 Meromorphic Functions 297
3.5.3 Locally Bounded Meromorphic Functions 302
3.5.4 Universal Denominators 307
3.5.5 Additional Material on Normality 311
3.5.6 Normalization 313
3.5.7 The Weak Normalization 316
3.5.8 Complementary Material on Meromorphic Functions 320
3.6 Local Bound of Volume of General Fibers 322
3.6.1 Local Blowing Up 322
3.6.2 The Theorem 326
3.7 Direct Image and Enclosure 330
3.7.1 Holomorphic Mappings with Values in a TVS 330
3.7.2 Reduced Multigraphs in a Sequentially Complete TVS ... 333
3.7.3 The Direct Image Theorem 337
3.7.4 Theorem on Encloseability 343
3.8 Holomorphic Convexity: The Quotient Theorem 355
3.8.1 Dirac Mapping 355
3.8.2 Holomorphically Convex Spaces 356
3.8.3 Stein Spaces and the Remmert Reduction 359
3.8.4 The Quotient Theorem of H. Cartan 362
3.9 Notes on This Chapter 365
3.9.1 Transversality and the Zariski Tangent Cone 365
3.9.2 Algebraic Cones 366
3.9.3 The Theorem of P. Lelong, Canonical Topology, Modifications and Blowups 366
3.9.4 Normalization 366
3.9.5 Weak Normalization 367
3.9.6 Bound of Volume of General Fibers 367
3.9.7 Direct Image and Encloseability: Holomorphic Convexity 367
3.9.8 Quotient Theorem 368

4 Families of Cycles in Complex Geometry 369
4.1 Families of Cycles 369
4.1.1 Cycles 369
4.1.2 Elementary Operations on Cycles 373
4.1.3 Functorial Properties 376

4.2 Continuous Families of Cycles 385
4.2.1 Scales 385
4.2.2 Topology of $\mathcal{C}_n^{loc}(M)$ and of $\mathcal{C}_n(M)$ 387
4.2.3 Functions Defined by Integration 398
4.2.4 $\mathcal{C}_n^{loc}(M)$ and $\mathcal{C}_n(M)$ are Second Countable 406
4.2.5 Continuity of Direct Image Maps 412
4.2.6 Integration of Cohomology Classes: Topological Case 419
4.2.7 Compactness and the Theorem of E. Bishop 428
4.2.8 Cycles as Currents 445
4.3 Analytic Families of Cycles 450
4.3.1 Basic Definitions 450
4.3.2 Multiplicity of a Point in a Cycle 457
4.3.3 Graph of an Analytic Family of Cycles 458
4.3.4 The Case of a Normal Parameter Space 460
4.3.5 Stability of Analytic Families by Direct Images 466
4.4 Fundamental Counterexample 471
4.4.1 What Does Not Work! 471
4.4.2 Example 472
4.5 Characterization of Isotropic Morphisms: Applications 476
4.5.1 Isotropic Morphisms 476
4.5.2 Integration of Cohomology Classes 480
4.6 Finiteness of the Space of Cycles: Applications 484
4.6.1 Finiteness Theorem 484
4.6.2 Some Consequences 485
4.7 Theorem on Connectedness 487
4.7.1 Number of Irreducible Components 487
4.7.2 Connected Cycles 488
4.8 Relative Cycles 491
4.8.1 Preliminaries 491
4.8.2 The Space of Cycles Relative to a Morphism 492
4.9 Fibers of a Proper Meromorphic Mapping 496
4.9.1 The Case of a Proper Holomorphic Map 496
4.9.2 The Case of a Proper Meromorphic Map 500
4.9.3 Almost Holomorphic Mappings 501
4.10 Analytic Families of Holomorphic Mappings 503
4.11 Appendix I: Complexification 508
4.11.1 Conjugation on a Complex Vector Space 508
4.11.2 Complexification of a Real Vector Space 510
4.11.3 The Complex Case 512
4.11.4 Orientation of a Complex Vector Space 514

4.11.5 The Space $\Lambda_{\mathbb{R}}^{p,p}(E)$ 515
4.11.6 Positivity in the Sense of P. Lelong 519
4.12 Appendix II: Locally Convex Topological Vector Spaces 521

Bibliography 525

Index 529