CONTENTS

1 **Introduction** 1
1.1 Problem Formulation 1

Part I Discrete Time Models

2 **The Binomial Model** 7
2.1 The One Period Model 7
2.1.1 Model Description 7
2.1.2 Portfolios and Arbitrage 8
2.1.3 Contingent Claims 11
2.1.4 Risk Neutral Valuation 13
2.2 The Multiperiod Model 16
2.2.1 Portfolios and Arbitrage 16
2.2.2 Contingent Claims 19
2.3 Exercises 26
2.4 Notes 26

3 **A More General One Period Model** 27
3.1 The Model 27
3.2 Absence of Arbitrage 28
3.3 Martingale Measures 33
3.4 Martingale Pricing 34
3.5 Completeness 36
3.6 Stochastic Discount Factors 38
3.7 Exercises 39

Part II Stochastic Calculus

4 **Stochastic Integrals** 43
4.1 Introduction 43
4.1.1 The Wiener Process 43
4.2 Information 45
4.3 Stochastic Integrals 47
4.4 Martingales 49
4.5 Stochastic Calculus and the Itô Formula 51
4.6 Examples 56
4.7 The Multidimensional Itô Formula 60
4.8 Correlated Wiener Processes 62
4.9 Exercises 64
4.10 Notes 66

5 **Stochastic Differential Equations** 67
5.1 Stochastic Differential Equations 67
5.2 Geometric Brownian Motion 68
5.3 The Linear SDE 71
5.4 The Infinitesimal Operator 72
5.5 Partial Differential Equations 73
5.6 The Kolmogorov Equations 76
5.7 Exercises 79
5.8 Notes 83

Part III Arbitrage Theory

6 **Portfolio Dynamics** 87
6.1 Introduction 87
6.2 Self-financing Portfolios in Discrete Time 87
6.2.1 Basic Definitions 87
6.2.2 Self-financing Portfolios 88
6.2.3 The Cumulative Dividend Process 90
6.3 Self-financing Portfolios in Continuous Time 91
6.4 Portfolio Weights 93

7 **Arbitrage Pricing** 95
7.1 Introduction 95
7.2 More on the Bank Account 96
7.3 Contingent Claims and Arbitrage 98
7.4 The Black–Scholes Equation 103
7.5 Risk Neutral Valuation 106
7.6 The Black–Scholes Formula 108
7.7 Forward and Futures Contracts 110
7.7.1 Forward Contracts 110
7.7.2 Futures Contracts and the Black Formula 111
7.8 Volatility 112
7.8.1 Historic Volatility 113
7.8.2 Implied Volatility 113
7.9 American Options 114
7.10 Exercises 116
7.11 Notes 118

8 **Completeness and Hedging** 119
8.1 Introduction 119
8.2 Completeness in the Black–Scholes Model 120
8.3 Completeness—Absence of Arbitrage 125
8.4 Exercises 126
8.5 Notes 127

9 A Primer on Incomplete Markets 128
9.1 Introduction 128
9.2 A Scalar Non-priced Underlying Asset 128
9.3 Summing Up 135
9.4 Exercises 137
9.5 Notes 137

10 Parity Relations and Delta Hedging 138
10.1 Parity Relations 138
10.2 The Greeks 140
10.3 Delta and Gamma Hedging 144
10.4 Exercises 147

11 The Martingale Approach to Arbitrage Theory 150
11.1 The Case of Zero Interest Rate 151
11.2 Absence of Arbitrage and Martingale Measures 153
11.3 A Rough Sketch of the Proof 154
11.3.1 Existence of an EMM Implies Absence of Arbitrage 154
11.3.2 Absence of Arbitrage Implies Existence of an EMM 155
11.4 The General Case 159
11.5 Completeness 162
11.6 Pricing Contingent Claims 164
11.7 Pricing by Replication 165
11.8 Stochastic Discount Factors 166
11.9 Summary for the Working Economist 167
11.10 Notes 170

12 The Mathematics of the Martingale Approach 171
12.1 Stochastic Integral Representations 171
12.2 The Girsanov Theorem: Heuristics 174
12.3 The Girsanov Theorem 176
12.4 The Converse of the Girsanov Theorem 179
12.5 Girsanov Transformations and Stochastic Differentials 180
12.6 Maximum Likelihood Estimation 181
12.7 Exercises 183
12.8 Notes 184

13 Black–Scholes from a Martingale Point of View 185
13.1 Absence of Arbitrage 185
13.2 Pricing 187
13.3 Completeness 187

14 Multidimensional Models: Martingale Approach 191
14.1 Absence of Arbitrage 192
14.2 Completeness 193
14.3 Hedging 194

14.4 Pricing 196
14.5 Markovian Models and PDEs 197
14.6 Market Prices of Risk 198
14.7 The Stochastic Discount Factor 199
14.8 The Hansen–Jagannathan Bounds 200
14.9 Exercises 201
14.10 Notes 201

15 Change of Numeraire 202
15.1 Introduction 202
15.2 Generalities 202
15.3 Changing the Numeraire 204
15.4 Some Examples 207
15.5 Forward Measures 210
15.5.1 Using the T-bond as Numeraire 211
15.6 A General Option Pricing Formula 212
15.6.1 General Theory 212
15.6.2 The Case of Deterministic Volatility 213
15.7 The Numeraire Portfolio 215
15.7.1 General Theory 215
15.7.2 The Objective Measure P as a Martingale Measure 216
15.8 Exercises 217
15.9 Notes 217

16 Dividends 219
16.1 Discrete Dividends 219
16.1.1 Dividend Structure 219
16.1.2 The Price Structure 220
16.1.3 A Black–Scholes Model with a Discrete Dividend 221
16.1.4 Option Pricing 222
16.1.5 Risk Neutral Valuation 223
16.1.6 An Example 224
16.2 Continuous Dividends I: Classical Methods 226
16.2.1 Continuous Dividend Yield 226
16.3 Continuous Dividends II: Martingale Methods 229
16.3.1 The Bank Account as Numeraire 230
16.3.2 Continuous Dividend Yield Revisited 231
16.3.3 An Arbitrary Numeraire 232
16.4 Exercises 235

17 Forward and Futures Contracts 236
17.1 Forward Contracts 236
17.2 Futures Contracts 238
17.3 Futures Options and Black-76 241
17.3.1 Generalities 241
17.3.2 The Black-76 Formula 241

17.4 Exercises 243
17.5 Notes 243

18 Currency Derivatives 244
18.1 Pure Currency Contracts 244
18.2 The Martingale Approach 247
18.3 Domestic and Foreign Equity Markets 250
18.4 An Extended Black–Scholes Model 252
18.5 The Siegel Paradox 253
18.6 Exercises 255
18.7 Notes 256

19 Bonds and Interest Rates 257
19.1 Zero Coupon Bonds 257
19.2 Interest Rates 258
19.2.1 Definitions 258
19.2.2 Relations between $df(t,T)$, $dp(t,T)$, and $dr(t)$ 260
19.2.3 An Expectation Hypothesis 263
19.2.4 An Alternative View of the Money Account 264
19.3 Coupon Bonds, Swaps, and Yields 265
19.3.1 Fixed Coupon Bonds 266
19.3.2 Floating Rate Bonds 266
19.3.3 Interest Rate Swaps 267
19.3.4 Yield and Duration 269
19.4 Exercises 270
19.5 Notes 271

20 Short Rate Models 272
20.1 Generalities 272
20.2 The Term Structure Equation 274
20.3 Martingale Analysis 276
20.4 Exercises 278
20.5 Notes 279

21 Martingale Models for the Short Rate 280
21.1 Q-Dynamics 280
21.2 Properties of the Short Rate Models 281
21.2.1 Models with Linear Dynamics 281
21.2.2 Models with Mean Reversion 282
21.2.3 Lognormal Models 283
21.2.4 Square Root Models 283
21.3 Inversion of the Yield Curve 284
21.4 Affine Term Structures 285
21.4.1 Definition and Existence 285
21.5 Analytical Results for Some Standard Models 288

21.5.1 The Vasiček Model 288
21.5.2 The Ho–Lee Model 289
21.5.3 The CIR Model 290
21.5.4 The Hull–White Model 290
21.6 Bond Options in the Hull–White Model 292
21.7 Exercises 293
21.8 Notes 295

22 Forward Rate Models 296
22.1 The Heath–Jarrow–Morton Framework 296
22.2 Martingale Modeling 298
22.3 The General Gaussian Model 300
22.4 The Musiela Parameterization 301
22.5 Exercises 302
22.6 Notes 304

23 LIBOR Market Models 305
23.1 Caps: Definition and Market Practice 306
23.2 The LIBOR Market Model 308
23.3 Pricing Caps in the LIBOR Model 309
23.4 Terminal Measure Dynamics and Existence 310
23.5 Calibration and Simulation 313
23.6 The Discrete Savings Account 314
23.7 Notes 316

24 Potentials and Positive Interest 317
24.1 Generalities 317
24.2 The Flesaker–Hughston Framework 318
24.3 Changing Base Measure 321
24.4 Decomposition of a Potential 322
24.5 The Markov Potential Approach of Rogers 323
24.6 Exercises 328
24.7 Notes 329

Part IV Optimal Control and Investment Theory

25 Stochastic Optimal Control 333
25.1 An Example 333
25.2 The Formal Problem 334
25.3 Embedding the Problem 337
25.4 Time Consistency and the Bellman Principle 339
25.5 Deriving the Hamilton–Jacobi–Bellman Equation 340
25.6 Handling the HJB Equation 345
25.7 The Linear Regulator 347
25.8 Exercises 349
25.9 Notes 350

26 Optimal Consumption and Investment 351
26.1 A Generalization 351
26.2 Optimal Consumption and Investment 352
26.3 The Mutual Fund Theorems 355
26.3.1 The Case with No Risk Free Asset 355
26.3.2 The Case with a Risk Free Asset 359
26.4 Exercises 361
26.5 Notes 363

27 The Martingale Approach to Optimal Investment 364
27.1 Generalities 364
27.2 The Basic Idea 365
27.3 The Optimal Terminal Wealth 366
27.4 The Optimal Wealth Process 367
27.5 The Optimal Portfolio 368
27.6 Log Utility 369
27.6.1 The Optimal Terminal Wealth 369
27.6.2 The Optimal Wealth Process 369
27.6.3 The Optimal Portfolio 370
27.7 Other Utility Functions 371
27.8 Optimal Consumption Problems 371
27.9 Exercises 374
27.10 Notes 375

28 Optimal Stopping Theory and American Options 376
28.1 Introduction 376
28.2 Generalities 376
28.3 Some Simple Results 377
28.4 Discrete Time 378
28.4.1 The General Case 378
28.4.2 Markovian Models 382
28.4.3 Infinite Horizon 384
28.5 Continuous Time 386
28.5.1 General Theory 386
28.5.2 Diffusion Models 387
28.5.3 Connections to the General Theory 392
28.6 American Options 392
28.6.1 The American Call without Dividends 392
28.6.2 The American Put Option 392
28.6.3 The Perpetual American Put 394
28.7 Exercises 395
28.8 Notes 395

Part V Incomplete Markets

29 Incomplete Markets 399
29.1 Introduction 399
29.2 A Markov Factor Model 400
29.3 The Independent Factor Markov Model 401
29.3.1 Absence of Arbitrage 402
29.3.2 Incompleteness 402
29.4 Methods to Handle Market Incompleteness 403
29.5 Notes 404

30 The Esscher Transform and the Minimal Martingale Measure 405
30.1 The Esscher Transform 405
30.1.1 The Standard Esscher Transform 405
30.1.2 The Generalized Esscher Transform 406
30.1.3 The Markov Factor Model 408
30.1.4 The Independent Factor Markov Model 408
30.2 The Minimal Martingale Measure 409
30.2.1 Definition and Existence 410
30.2.2 Basic Properties of Q^M 411
30.2.3 Economic Interpretation of Q^M 411
30.3 Notes 412

31 Minimizing f-Divergence 413
31.1 Definition and Basic Properties 413
31.2 Minimal Reverse Entropy 414
31.3 Minimal Entropy in a Factor Model 415
31.4 Duality 418
31.4.1 Utility Maximization of Financial Derivatives 419
31.4.2 Minimax Measures 421
31.4.3 Log Utility 423
31.4.4 Exponential Utility 423
31.5 Notes 425

32 Portfolio Optimization in Incomplete Markets 426
32.1 Setup 426
32.2 The Complete Market Case 427
32.3 Incomplete Market, Finite Ω 428
32.4 Incomplete Market, General Ω 433
32.5 Notes 435

33 Utility Indifference Pricing and Other Topics 436
33.1 Global Indifference Pricing 436
33.2 Marginal Indifference Pricing 438
33.3 Hedging 439
33.4 Notes 440

34 Good Deal Bounds 441
34.1 General Ideas 441
34.2 The Model 443
34.3 The Good Deal Bounds 443
34.4 The Embedded Optimization Problem 445
34.5 Relations to the Minimal Martingale Measure 445
34.6 An Option with Basis Risk 446
34.7 Notes 448

Part VI Dynamic Equilibrium Theory

35 Equilibrium Theory: A Simple Production Model 451
35.1 The Model 451
35.2 Equilibrium 453
35.3 Introducing a Central Planner 455
35.4 Exercises 457
35.5 Notes 457

36 The Cox–Ingersoll–Ross Factor Model 458
36.1 The Model 458
36.1.1 Exogenous Objects 458
36.1.2 Endogenous Objects 458
36.1.3 Economic Agents 459
36.2 The Portfolio Problem 459
36.2.1 Portfolio Dynamics 460
36.2.2 The Control Problem and the HJB Equation 460
36.3 Equilibrium 461
36.4 The Short Rate and the Risk Premium for F 462
36.5 The Equilibrium Stochastic Discount Factor 462
36.6 Risk Neutral Valuation 464
36.7 Introducing a Central Planner 464
36.8 Exercises 465
36.9 Notes 466

37 The Cox–Ingersoll–Ross Interest Rate Model 467
37.1 Exercises 469
37.2 Notes 469

38 Endowment Equilibrium: Unit Net Supply 470
38.1 The Model 470
38.1.1 Exogenous Objects 470
38.1.2 Endogenous Objects 470
38.1.3 Economic Agents 471
38.1.4 Equilibrium Conditions 472

38.2 The Martingale Approach 472
38.2.1 The Control Problem 472
38.2.2 Equilibrium 473
38.2.3 Log Utility 474
38.3 Extending the Model 475
38.3.1 The General Scalar Case 475
38.3.2 A Factor Model 476
38.4 Several Endowment Processes 478
38.5 Exercises 479
38.6 Notes 480

Appendices

A Measure and Integration 481
A.1 Sets and Mappings 481
A.2 Measures and Sigma-Algebras 483
A.3 Integration 485
A.4 Sigma-Algebras and Partitions 489
A.5 Sets of Measure Zero 490
A.6 The L^p Spaces 491
A.7 Hilbert Spaces 492
A.8 Sigma-Algebras and Generators 495
A.9 Product Measures 499
A.10 The Lebesgue Integral 499
A.11 The Radon–Nikodym Theorem 500
A.12 Exercises 504
A.13 Notes 506

B Probability Theory 507
B.1 Random Variables and Processes 507
B.2 Partitions and Information 510
B.3 Sigma-Algebras and Information 511
B.4 Independence 514
B.5 Conditional Expectations 516
B.6 Equivalent Probability Measures 522
B.7 Exercises 524
B.8 Notes 525

C Martingales and Stopping Times 526
C.1 Martingales 526
C.2 Discrete Stochastic Integrals 529
C.3 Likelihood Processes 530
C.4 Stopping Times 530
C.5 Exercises 533

D Convex Duality 536
D.1 Conjugate Functions 536
D.2 Lagrange Functions and Saddle Points 536
D.3 An Envelope Theorem 537

References 539

Index 549