Introductory Lectures on Equivariant Cohomology

Loring W. Tu

With Appendices by Loring W. Tu and Alberto Arabia

PRINCETON UNIVERSITY PRESS PRINCETON AND OXFORD 2020

Contents

List of Figures		xv	
Preface			xvii
Acknowledgments		xix	
Ι	$\mathbf{E}\mathbf{q}$	uivariant Cohomology in the Continuous Category	1
1	Ove	erview	5
	1.1	Actions of a Group	5
	1.2	Orbits, Stabilizers, and Fixed Point Sets	6
	1.3	Homogeneous Spaces	7
	1.4	Equivariant Cohomology	8
		Problems	10
2	Homotopy Groups and CW Complexes		11
	2.1	Homotopy Groups	11
	2.2	Fiber Bundles	13
	2.3	Homotopy Exact Sequence of a Fiber Bundle	14
	2.4	Attaching Cells	15
	2.5	CW Complexes	16
	2.6	Manifolds and CW Complexes	17
	2.7	The Infinite Sphere	18
		Problems	19
3	Principal Bundles		21
	3.1	Principal Bundles	21
	3.2	The Pullback of a Fiber Bundle	24
		Problems	26
4	Homotopy Quotients and Equivariant Cohomology		29
	4.1	A First Candidate for Equivariant Cohomology	29
	4.2	Homotopy Quotients	30
	4.3	Cartan's Mixing Space and Cartan's Mixing Diagram	31
	4.4	Equivariant Cohomology Is Well-Defined	35

	4.5	Algebraic Structure of Equivariant Cohomology Problems	36 37
5	Uni	versal Bundles and Classifying Spaces	39
	5.1	Universal Bundles	39
	5.2	Uniqueness of a CW Classifying Space	41
	5.3	Milnor's Construction	41
	5.4	Equivariant Cohomology of a Point	43
6	\mathbf{Spe}	ectral Sequences	45
	6.1	Leray's Theorem	45
	6.2	Leray's Theorem on the Product Structure	49
	6.3	Example: The Cohomology of $\mathbb{C}P^2$	50
		Problems	54
7	Eqι	ivariant Cohomology of S^2 Under Rotation	57
	7.1	Homotopy Quotient as a Fiber Bundle	57
	7.2	Equivariant Cohomology of S^2 Under Rotation	57
		Problems	59
8	ΑŢ	Iniversal Bundle for a Compact Lie Group	61
	8.1	The Stiefel Variety	61
	8.2	A Principal $O(k)$ -Bundle	62
	8.3	Homotopy Groups of a Stiefel Variety	63
	8.4	Closed Subgroups of a Lie Group	64
	8.5	Universal Bundle for a Compact Lie Group	65
	8.6	Universal Bundle for a Direct Product	65
	8.7	Infinite-Dimensional Manifolds	66
		Problems	67
9	Gei	neral Properties of Equivariant Cohomology	69
	9.1	Functorial Properties	69
	9.2	Free Actions	72
	9.3	Coefficient Ring of Equivariant Cohomology	73
	9.4	Equivariant Cohomology of a Disjoint Union	74
		Problems	75
11	Di	fferential Geometry of a Principal Bundle	77
10) The	e Lie Derivative and Interior Multiplication	81
	10.1	The Lie Derivative of a Vector Field	81
	10.2	2 The Lie Derivative of a Differential Form	82
	10.3	Interior Multiplication	84

Problems	85
11 Fundamental Vector Fields	87
11.1 Fundamental Vector Fields	87
11.2 Zeros of a Fundamental Vector Field	90
11.3 Vertical Vectors on a Principal Bundle	92
11.4 Translate of a Fundamental Vector Field	93
11.5 The Lie Bracket of Fundamental Vector Fields	94
Problems	95
12 Basic Forms	97
12.1 Basic Forms on \mathbb{R}^2	97
12.2 Invariant Forms	98
12.3 Horizontal Forms	99
12.4 Basic Forms	100
Problems	101
13 Integration on a Compact Connected Lie Group	103
13.1 Bi-Invariant Forms on a Compact Connected Lie Group	103
13.2 Integration over a Compact Connected Lie Group	105
13.3 Invariance of the Integral	106
13.4 The Pullback of an Integral	109
13.5 Differentiation Under the Integral Sign	110
13.6 Cohomology Does Not Commute with Invariants	113
Problems	114
14 Vector-Valued Forms	115
14.1 Vector-Valued Forms	115
14.2 Lie Algebra Valued Forms	116
14.3 Matrix-Valued Forms	118
Problems	119
15 The Maurer–Cartan Form	121
15.1 The Lie Algebra \mathfrak{q} of a Lie Group and Its Dual \mathfrak{q}^{\vee}	121
15.2 Maurer-Cartan Equation with Respect to a Basis	123
15.3 The Maurer–Cartan Form	124
16 Connections on a Principal Bundle	127
16.1 Maps of Vector Bundles	127
16.2 Vertical and Horizontal Subbundles	128
16.3 Connections on a Principal Bundle	129
16.4 The Maurer–Cartan Form Is a Connection	131
16.5 Existence of a Connection on a Principal Bundle	132

17 Curvature on a Principal Bundle	135
17.1 Curvature	135
17.2 Properties of Curvature	136
III The Cartan Model	141
18 Differential Graded Algebras	145
18.1 Differential Graded Algebras	145
18.2 Tensor Product of Differential Graded Algebras	147
18.3 The Basic Subcomplex of a \mathfrak{g} -Differential Graded Algebra	148
Problems	150
19 The Weil Algebra and the Weil Model	151
19.1 The Weil Algebra and the Weil Map	151
19.2 The Weil Map Relative to a Basis	152
19.3 The Weil Algebra as a \mathfrak{g} -DGA	153
19.4 The Cohomology of the Weil Algebra	156
19.5 An Algebraic Model for the Universal Bundle	158
19.6 An Algebraic Model for the Homotopy Quotient	159
19.7 Functoriality of the Weil Model	159
Problems	100
20 Circle Actions	161
20.1 The Weil Algebra for a Circle Action	161
20.2 The Weil Model for a Circle Action	161
20.3 The Cartan Model for a Circle Action	163
20.4 The Cartan Differential for a Circle Action	164
20.5 Example: The Action of a Circle on a Point	105
21 The Cartan Model in General	167
21.1 The Weil–Cartan Isomorphism	167
21.2 The Cartan Differential	171
21.3 Intrinsic Description of the Cartan Differential	172
21.4 Pullback of Equivariant Forms	174
21.5 The Equivariant de Rham Theorem	175
21.6 Equivariant Forms for a Torus Action	175
21.7 Example: The Action of a Torus on a Point	170
21.8 Equivariantly Closed Extensions Problems	170
r topients	1(9
22 Outline of a Proof of the Equivariant de Rham Theorem	181
22.1 The Cohomology of the Base	181
22.2 Equivariant de Rham Theorem for a Free Action	183

22.3 Equivariant de Rham Theorem in General 22.4 Cohomology of a Classifying Space	$183\\184$
IV Borel Localization	187
 23 Localization in Algebra 23.1 Localization with Respect to a Variable 23.2 Localization and Torsion 23.3 Antiderivations Under Localization 23.4 Localization and Exactness Problems 	189 189 192 192 193 196
 24 Free and Locally Free Actions 24.1 Equivariant Cohomology of a Free Action 24.2 Locally Free Actions 24.3 Equivariant Cohomology of a Locally Free Circle Action Problems 	197 197 198 202 204
 25 The Topology of a Group Action 25.1 Smooth Actions of a Compact Lie Group 25.2 Equivariant Vector Bundles 25.3 The Normal Bundle of a Submanifold 25.4 Equivariant Tubular Neighborhoods 25.5 Equivariant Mayer–Vietoris Sequence 	205 205 206 207 209 211
26 Borel Localization for a Circle Action 26.1 Borel Localization 26.2 Example: The Ring Structure of $H_{S^1}^*(S^2)$ Problems	213 213 216 219
V The Equivariant Localization Formula	22 1
 27 A Crash Course in Representation Theory 27.1 Representations of a Group 27.2 Local Data at a Fixed Point Problems 	223 223 225 227
 28 Integration of Equivariant Forms 28.1 Manifolds with Boundary 28.2 Integration of Equivariant Forms 28.3 Stokes's Theorem for Equivariant Integration 28.4 Integration of Equivariant Forms for a Circle Action 	229 229 230 231 232

Problems	232
29 Rationale for a Localization Formula	233
29.1 Circle Actions with Finitely Many Fixed Points	233
29.2 The Spherical Blow-Up	235
Problems	238
30 Localization Formulas	239
30.1 Equivariant Localization Formula for a Circle Action	239
30.2 Application: The Area of a Sphere	241
30.3 Equivariant Characteristic Classes	242
30.4 Localization Formula for a Torus Action	243
Problems	244
31 Proof of the Localization Formula for a Circle Action	245
31.1 On the Spherical Blow-Up	245
31.2 Surface Area of a Sphere	249
32 Some Applications	253
32.1 Integration of Invariant Forms	253
32.2 Rational Cohomology of a Homogeneous Space	253
32.3 Topological Invariants of a Homogeneous Space	254
32.4 Symplectic Geometry and Classical Mechanics	255
32.5 Stationary Phase Approximation	256
32.6 Exponents at Fixed Points	256
32.7 Gysin Maps	257
Appendices	
A. Proof of the Equivariant de Rham Theorem	
by Loring W. Tu and Alberto Arabia	261
A.1 The Weil Algebra	262
A.2 The Well Map	263
A.3 Conomology of Basic Subcomplexes	263
A.4 Cartan's Operator A	205
A.5 A Degree-Lowering Property	200
A 7 Equivariant de Rham Theorem for a Free Action	208
A 8 Approximation Theorems	200 260
A.9 Proof of the Equivariant de Rham Theorem in Ceneral	209 260
A.10 Approximations of EG	203
A.11 Approximations of the Homotopy Quotient M_C	272
A.12 A Spectral Sequence for the Cartan Model	274

CONTENTS

CONTENTS

A.13 Ordinary Cohomology and the Cohomology of the Cartan Model	275
B. A Comparison Theorem for Spectral Sequences by Alberto Arabia	279
C. Commutativity of Cohomology with Invariants by Alberto Arabia	283
Hints and Solutions to Selected End-of-Section Problems	289
List of Notations	297
Bibliography	303
Index	309