Contents

Preface *page* ix

1 Static optimization 1
1.1 Unconstrained optimization, concave and convex functions 1
1.2 Optimization under equality constraints: the method of Lagrange 20
1.3 Comparative statics 43
1.4 Optimization under inequality constraints: nonlinear programming 52
1.5 Economic applications of nonlinear programming 67
1.6 The special case of linear programming 70
Appendix 74
Exercises 79

2 Ordinary differential equations 87
2.1 Introduction 87
2.2 Definitions and fundamental results 88
2.3 First-order differential equations 91
2.4 Systems of linear FODE with constant coefficients 95
2.5 Systems of two nonlinear FODE 100
Appendix 111
Exercises 113

3 Introduction to dynamic optimization 117
3.1 Optimal borrowing 118
3.2 Fiscal policy 119
3.3 Suboptimal consumption path 120
3.4 Discounting and depreciation in continuous-time models 121
Exercises 124

4 The maximum principle 127
4.1 A simple control problem 127
4.2 Derivation of the maximum principle in discrete time 129
4.3 Numerical solution of an optimal control problem in continuous time 133

4.4 Phase diagram analysis of optimal control problems 137
4.5 Economic interpretation of the maximum principle 151
4.6 Necessity and sufficiency of the maximum principle 161
Exercises 165

5 The calculus of variations and dynamic programming 169
5.1 The calculus of variations 169
5.2 Dynamic programming: discrete-time, finite-horizon problems 173
5.3 Dynamic programming in continuous time 182
Exercises 184

6 The general constrained control problem 187
6.1 The set of admissible controls 187
6.2 Integral constraints 190
6.3 The maximum principle with equality constraints only 192
6.4 The maximum principle with inequality constraints 198
6.5 Necessity and sufficiency theorems: the case with inequality and equality constraints 210
6.6 Concluding notes 218
Exercises 218

7 Endpoint constraints and transversality conditions 221
7.1 Free-endpoint problems 222
7.2 Problems with free endpoint and a scrap value function 226
7.3 Lower bound constraints on endpoint 229
7.4 Problems with lower bound constraints on endpoint and a scrap value function 235
7.5 Free-terminal-time problems without a scrap value function 240
7.6 Free-terminal-time problems with a scrap value function 244
7.7 Other transversality conditions 247
7.8 A general formula for transversality conditions 248
7.9 Sufficiency theorems 251
7.10 A summary table of common transversality conditions 253
7.11 Control parameters 253
Exercises 259

8 Discontinuities in the optimal controls 263
8.1 A classical bang-bang example 263
8.2 The beekeeper's problem 267
8.3 One-sector optimal growth with reserves 274

8.4 Highest consumption path 277
8.5 Concluding comments 281
Exercises 282

9 Infinite-horizon problems 285
9.1 Optimality criteria 285
9.2 Necessary conditions 287
9.3 Sufficient conditions 288
9.4 Autonomous problems 289
9.5 Steady states in autonomous infinite-horizon problems 294
9.6 Further properties of autonomous infinite-horizon problems 298
Exercises 304

10 Three special topics 307
10.1 Problems with two-state variables 307
10.2 Trade in capital goods: jumps in the state variables 310
10.3 Constraints on the state variables 332
Exercises 342

Bibliography 345
Index 351