
Arnold Diffusion for Smooth Systems of Two and a Half Degrees of Freedom

Vadim | Kaloshin

Ke Zhang

PRINCETON UNIVERSITY PRESS
PRINCETON AND OXFORD
2020

Contents

Preface	xi
Acknowledgments	xiii
I Introduction and the general scheme	1
1 Introduction	3
1.1 Statement of the result	3
1.2 Scheme of diffusion	7
1.3 Three regimes of diffusion	11
1.4 The outline of the proof	12
1.5 Discussion	14
2 Forcing relation	17
2.1 Sufficient condition for Arnold diffusion	17
2.2 Diffusion mechanisms via forcing equivalence	18
2.3 Invariance under the symplectic coordinate changes	20
2.4 Normal hyperbolicity and Aubry-Mather type	22
3 Normal forms and cohomology classes at single resonances	24
3.1 Resonant component and non-degeneracy conditions	24
3.2 Normal form	26
3.3 The resonant component	29
4 Double resonance: geometric description	31
4.1 The slow system	31
4.2 Non-degeneracy conditions for the slow system	32
4.3 Normally hyperbolic cylinders	34
4.4 Local maps and global maps	36
5 Double resonance: forcing equivalence	39
5.1 Choice of cohomologies for the slow system	39
5.2 Aubry-Mather type at a double resonance	42
5.3 Connecting to Γ_{k_1, k_2} and $\Gamma_{k_1}^{SR}$	44
5.3.1 Connecting to the double resonance point	44
5.3.2 Connecting single and double resonance	45

5.4	Jump from non-simple homology to simple homology	49
5.5	Forcing equivalence at the double resonance	49
II	Forcing relation and Aubry-Mather type	53
6	Weak KAM theory and forcing equivalence	55
6.1	Periodic Tonelli Hamiltonians	55
6.2	Weak KAM solution	57
6.3	Pseudographs, Aubry, Mañé, and Mather sets	59
6.4	The dual setting, forward solutions	60
6.5	Peierls barrier, static classes, elementary solutions	62
6.6	The forcing relation	64
6.7	The Green bundles	65
7	Perturbative weak KAM theory	66
7.1	Semi-continuity	66
7.2	Continuity of the barrier function	68
7.3	Lipschitz estimates for nearly integrable systems	70
7.4	Estimates for nearly autonomous systems	71
8	Cohomology of Aubry-Mather type	77
8.1	Aubry-Mather type and diffusion mechanisms	77
8.2	Weak KAM solutions are unstable manifolds	83
8.3	Regularity of the barrier functions	86
8.4	Bifurcation type	88
III	Proving forcing equivalence	91
9	Aubry-Mather type at the single resonance	93
9.1	The single maximum case	93
9.2	Aubry-Mather type at single resonance	94
9.3	Bifurcations in the double maxima case	96
9.4	Hyperbolic coordinates	97
9.5	Normally hyperbolic invariant cylinder	100
9.6	Localization of the Aubry and Mañé sets	102
9.7	Genericity of the single-resonance conditions	103
10	Normally hyperbolic cylinders at double resonance	106
10.1	Normal form near the hyperbolic fixed point	107
10.2	Shil'nikov's boundary value problem	108
10.3	Properties of the local maps	110
10.4	Periodic orbits for the local and global maps	114
10.5	Normally hyperbolic invariant manifolds	118
10.6	Cyclic concatenations of simple geodesics	119

11 Aubry-Mather type at the double resonance	121
11.1 High-energy case	121
11.2 Simple non-critical case	125
11.3 Simple critical case	126
11.3.1 Proof of Aubry-Mather type using local coordinates . . .	126
11.3.2 Construction of the local coordinates	129
12 Forcing equivalence between kissing cylinders	133
12.1 Variational problem for the slow mechanical system	133
12.2 Variational problem for original coordinates	136
12.3 Scaling limit of the barrier function	139
12.4 The jump mechanism	140
IV Supplementary topics	145
13 Generic properties of mechanical systems on the two-torus	147
13.1 Generic properties of periodic orbits	147
13.2 Generic properties of minimal orbits	153
13.3 Non-degeneracy at high-energy	156
13.4 Unique hyperbolic minimizer at very high energy	158
13.5 Generic properties at the critical energy	160
14 Derivation of the slow mechanical system	162
14.1 Normal forms near maximal resonances	162
14.2 Affine coordinate change, rescaling, and energy reduction . . .	172
14.3 Variational properties of the coordinate changes	177
15 Variational aspects of the slow mechanical system	182
15.1 Relation between the minimal geodesics and the Aubry sets . .	182
15.2 Characterization of the channel and the Aubry sets	185
15.3 The width of the channel	188
15.4 The case $E = 0$	190
Appendix: Notations	195
References	199