

Contents

Part I THE ROOTS OF COMPUTABILITY THEORY

1	Introduction	3
1.1	Algorithms and Computation	3
1.1.1	The Intuitive Concept of the Algorithm and Computation ...	3
1.1.2	Algorithms and Computations Before the Twentieth Century	6
1.2	Chapter Summary	7
2	The Foundational Crisis of Mathematics	9
2.1	Crisis in Set Theory	9
2.1.1	Axiomatic Systems	9
2.1.2	Cantor's Naive Set Theory	13
2.1.3	Logical Paradoxes	17
2.2	Schools of Recovery	19
2.2.1	Slowdown and Revision	20
2.2.2	Intuitionism	20
2.2.3	Logicism	23
2.2.4	Formalism	26
2.3	Chapter Summary	29
3	Formalism	31
3.1	Formal Axiomatic Systems and Theories	31
3.1.1	What Is a Formal Axiomatic System?	31
3.1.2	The Notion of Truth	35
3.1.3	Interpretations and Models	40
3.2	Formalization of Logic, Arithmetic, and Set Theory	44
3.3	Chapter Summary	53
4	Hilbert's Attempt at Recovery	55
4.1	Hilbert's Program	55
4.1.1	Fundamental Problems of the Foundations of Mathematics	55

4.1.2	Hilbert's Program	59
4.2	The Fate of Hilbert's Program	60
4.2.1	Formalization of Mathematics: Formal Axiomatic System M	60
4.2.2	Decidability of M : Entscheidungsproblem	61
4.2.3	Completeness of M : Gödel's First Incompleteness Theorem	63
4.2.4	Consequences of the First Incompleteness Theorem	64
4.2.5	Consistency of M : Gödel's Second Incompleteness Theorem	66
4.2.6	Consequences of the Second Incompleteness Theorem	67
4.3	Legacy of Hilbert's Program	69
4.4	Chapter Summary	70
	Problems	71
	Bibliographic Notes	72

Part II CLASSICAL COMPUTABILITY THEORY

5	The Quest for a Formalization	77
5.1	What Is an Algorithm and What Do We Mean by Computation?	77
5.1.1	Intuition and Dilemmas	78
5.1.2	The Need for Formalization	79
5.2	Models of Computation	80
5.2.1	Modeling After Functions	80
5.2.2	Modeling After Humans	88
5.2.3	Modeling After Languages	90
5.2.4	Reasonable Models of Computation	95
5.3	Computability (Church-Turing) Thesis	96
5.3.1	History of the Thesis	96
5.3.2	The Thesis	97
5.3.3	Difficulties with Total Functions	99
5.3.4	Generalization to Partial Functions	102
5.3.5	Applications of the Thesis	106
5.4	Chapter Summary	106
	Problems	107
	Bibliographic Notes	109
6	The Turing Machine	111
6.1	Turing Machine	111
6.1.1	Basic Model	112
6.1.2	Generalized Models	117
6.1.3	Equivalence of Generalized and Basic Models	119
6.1.4	Reduced Model	123
6.1.5	Equivalence of Reduced and Basic Models	124
6.1.6	Use of Different Models	124
6.2	Universal Turing Machine	125
6.2.1	Coding and Enumeration of Turing Machines	125
6.2.2	The Existence of a Universal Turing Machine	127

6.2.3	The Importance of the Universal Turing Machine	129
6.2.4	Practical Consequences: Data vs. Instructions	129
6.2.5	Practical Consequences: General-Purpose Computer	129
6.2.6	Practical Consequences: Operating System	131
6.2.7	Practical Consequences: RAM Model of Computation	132
6.3	Use of a Turing Machine	135
6.3.1	Function Computation	135
6.3.2	Set Generation	137
6.3.3	Set Recognition	140
6.3.4	Generation vs. Recognition	143
6.3.5	The Standard Universes Σ^* and \mathbb{N}	146
6.3.6	Formal Languages vs. Sets of Natural Numbers	147
6.4	Chapter Summary	148
	Problems	149
	Bibliographic Notes	152
7	The First Basic Results	155
7.1	Some Basic Properties of Semi-decidable (C.E.) Sets	155
7.2	Padding Lemma and Index Sets	157
7.3	Parameter (s-m-n) Theorem	159
7.3.1	Deduction of the Theorem	160
7.4	Recursion (Fixed-Point) Theorem	161
7.4.1	Deduction of the Theorem	162
7.4.2	Interpretation of the Theorem	163
7.4.3	Fixed Points of Functions	164
7.4.4	Practical Consequences: Recursive Program Definition	165
7.4.5	Practical Consequences: Recursive Program Execution	166
7.4.6	Practical Consequences: Procedure Calls in General-Purpose Computers	169
7.5	Chapter Summary	171
	Problems	171
	Bibliographic Notes	173
8	Incomputable Problems	175
8.1	Problem Solving	175
8.1.1	Decision Problems and Other Kinds of Problems	176
8.1.2	Language of a Decision Problem	177
8.1.3	Subproblems of a Decision Problem	179
8.2	There Is an Incomputable Problem — Halting Problem	180
8.2.1	Consequences: The Basic Kinds of Decision Problems	183
8.2.2	Consequences: Complementary Sets and Decision Problems	185
8.2.3	Consequences: There Is an Incomputable Function	186
8.3	Some Other Incomputable Problems	186
8.3.1	Problems About Turing Machines	187
8.3.2	Post's Correspondence Problem	189

8.3.3	Problems About Algorithms and Computer Programs	189
8.3.4	Problems About Programming Languages and Grammars . .	191
8.3.5	Problems About Computable Functions	193
8.3.6	Problems from Number Theory	194
8.3.7	Problems from Algebra	194
8.3.8	Problems from Analysis	196
8.3.9	Problems from Topology	197
8.3.10	Problems from Mathematical Logic	198
8.3.11	Problems About Games	199
8.4	Can We Outwit Incomputable Problems?	201
8.5	Chapter Summary	203
	Problems	203
	Bibliographic Notes	204
9	Methods of Proving Incomputability	205
9.1	Proving by Diagonalization	205
9.1.1	Direct Diagonalization	205
9.1.2	Indirect Diagonalization	208
9.2	Proving by Reduction	210
9.2.1	Reductions in General	210
9.2.2	The m -Reduction	211
9.2.3	Undecidability and m -Reduction	213
9.2.4	The 1-Reduction	215
9.3	Proving by the Recursion Theorem	218
9.4	Proving by Rice's Theorem	219
9.4.1	Rice's Theorem for P.C. Functions	219
9.4.2	Rice's Theorem for Index Sets	220
9.4.3	Rice's Theorem for C.E. Sets	222
9.4.4	Consequences: Behavior of Abstract Computing Machines . .	223
9.5	Incomputability of Other Kinds of Problems	224
9.6	Chapter Summary	227
	Problems	228
	Bibliographic Notes	230
Part III RELATIVE COMPUTABILITY		
10	Computation with External Help	233
10.1	Turing Machines with Oracles	233
10.1.1	Turing's Idea of Oracular Help	234
10.1.2	The Oracle Turing Machine (o -TM)	237
10.1.3	Some Basic Properties of o -TMs	239
10.1.4	Coding and Enumeration of o -TMs	240
10.2	Computation with Oracles	242
10.2.1	Generalization of Classical Definitions	242
10.2.2	Convention: The Universe \mathbb{N} and Single-Argument Functions	245

10.3 Other Ways to Make External Help Available	245
10.4 Relative Computability Thesis	246
10.5 Practical Consequences: σ -TM with a Database or Network	246
10.6 Practical Consequences: Online and Offline Computation	247
10.7 Chapter Summary	248
Bibliographic Notes	249
11 Degrees of Unsolvability	251
11.1 Turing Reduction	251
11.1.1 Turing Reduction of a Computational Problem	252
11.1.2 Some Basic Properties of the Turing Reduction	253
11.2 Turing Degrees	256
11.3 Chapter Summary	259
Problems	260
Bibliographic Notes	261
12 The Turing Hierarchy of Unsolvability	263
12.1 The Perplexities of Unsolvability	263
12.2 The Turing Jump	264
12.2.1 Properties of the Turing Jump of a Set	265
12.3 Hierarchies of T -Degrees	267
12.3.1 The Jump Hierarchy	268
12.4 Chapter Summary	270
Problems	270
Bibliographic Notes	271
13 The Class \mathcal{D} of Degrees of Unsolvability	273
13.1 The Structure $(\mathcal{D}, \leq, ')$	273
13.2 Some Basic Properties of $(\mathcal{D}, \leq, ')$	275
13.2.1 Cardinality of Degrees and of the Class \mathcal{D}	275
13.2.2 The Class \mathcal{D} as a Mathematical Structure	276
13.2.3 Intermediate T -Degrees	281
13.2.4 Cones	282
13.2.5 Minimal T -Degrees	284
13.3 Chapter Summary	285
Problems	285
Bibliographic Notes	286
14 C.E. Degrees and the Priority Method	287
14.1 C.E. Turing Degrees	287
14.2 Post's Problem	288
14.2.1 Post's Attempt at a Solution to Post's Problem	289
14.3 The Priority Method and Priority Arguments	292
14.3.1 The Priority Method in General	292
14.3.2 The Friedberg-Muchnik Solution to Post's Problem	296
14.3.3 Priority Arguments	297

14.4	Some Properties of C.E. Degrees	297
14.5	Chapter Summary	298
	Problems	298
	Bibliographic Notes	299
15	The Arithmetical Hierarchy	301
15.1	Decidability of Relations	301
15.2	The Arithmetical Hierarchy	302
15.3	The Link with the Jump Hierarchy	306
15.4	Practical Consequences: Proving Incomputability	308
15.5	Chapter Summary	310
	Problems	310
	Bibliographic Notes	311
Part IV BACK TO THE ROOTS		
16	Computability (Church-Turing) Thesis Revisited	315
16.1	Introduction	315
16.2	The Intuitive Understanding of the Notion of a “Procedure”	316
16.3	Toward the Thesis	317
16.3.1	Gödel	317
16.3.2	Church	319
16.3.3	Kleene	323
16.3.4	Rosser	323
16.3.5	Post	324
16.3.6	Turing	326
16.4	Church-Turing Thesis	337
16.4.1	Differences Between Church’s and Turing’s Theses	337
16.4.2	The Church-Turing Thesis	338
16.4.3	Justifications of the Church-Turing Thesis	339
16.4.4	Provability of the Church-Turing Thesis	343
16.5	Résumé and Warnings	345
16.6	New Questions About the Church-Turing Thesis	347
16.6.1	Original CTT	347
16.6.2	Algorithmic Versions of CTT	347
16.6.3	Complexity-Theoretic Versions of CTT	348
16.6.4	Physical Versions of CTT	349
16.6.5	Hypercomputing?	353
	Bibliographic Notes	357
17	Further Reading	359
A	Mathematical Background	363
B	Notation Index	371

Contents	xxi
Glossary	377
References	397
Index	409