
Contents

List of Figures	xiii
Acknowledgments	xv
1 Introduction	1
1.1 Basic notions in general relativity	1
1.1.1 Spacetime and causality	1
1.1.2 The initial value formulation for Einstein equations	2
1.1.3 Special solutions	3
1.1.4 Stability of Minkowski space	10
1.1.5 Cosmic censorship	11
1.2 Stability of Kerr conjecture	13
1.2.1 Formal mode analysis	15
1.2.2 Vectorfield method	16
1.3 Nonlinear stability of Schwarzschild under polarized perturbations	17
1.3.1 Bare-bones version of our theorem	17
1.3.2 Linear stability of the Schwarzschild spacetime	17
1.3.3 Main ideas in the proof of Theorem 1.6	18
1.3.4 Beyond polarization	21
1.3.5 Note added in proof	22
1.4 Organization	22
2 Preliminaries	24
2.1 Axially symmetric polarized spacetimes	24
2.1.1 Axial symmetry	24
2.1.2 \mathbb{Z} -frames	25
2.1.3 Axis of symmetry	26
2.1.4 \mathbb{Z} -polarized S -surfaces	28
2.1.5 Invariant S -foliations	45
2.1.6 Schwarzschild spacetime	50
2.2 Main equations	51
2.2.1 Main equations for general S -foliations	51
2.2.2 Null Bianchi identities	54
2.2.3 Hawking mass	56
2.2.4 Outgoing geodesic foliations	57
2.2.5 Additional equations	70
2.2.6 Ingoing geodesic foliation	71
2.2.7 Adapted coordinates systems	71
2.3 Perturbations of Schwarzschild and invariant quantities	78
2.3.1 Null frame transformations	78
2.3.2 Schematic notation Γ_g and Γ_b	82

2.3.3	The invariant quantity q	83
2.3.4	Several identities for q	84
2.4	Invariant wave equations	84
2.4.1	Preliminaries	85
2.4.2	Wave equations for α , $\underline{\alpha}$, and q	87
3	Main Theorem	89
3.1	General covariant modulated admissible spacetimes	89
3.1.1	Initial data layer	89
3.1.2	Main definition	91
3.1.3	Renormalized curvature components and Ricci coefficients .	95
3.2	Main norms	96
3.2.1	Main norms in $(^{ext})\mathcal{M}$	96
3.2.2	Main norms in $(^{int})\mathcal{M}$	99
3.2.3	Combined norms	100
3.2.4	Initial layer norm	100
3.3	Main theorem	101
3.3.1	Smallness constants	101
3.3.2	Statement of the main theorem	102
3.4	Bootstrap assumptions and first consequences	105
3.4.1	Main bootstrap assumptions	105
3.4.2	Control of the initial data	105
3.4.3	Control of averages and of the Hawking mass	106
3.4.4	Control of coordinates system	107
3.4.5	Pointwise bounds for higher order derivatives	109
3.4.6	Construction of a second frame in $(^{ext})\mathcal{M}$	109
3.5	Global null frames	111
3.5.1	Extension of frames	111
3.5.2	Construction of the first global frame	112
3.5.3	Construction of the second global frame	113
3.6	Proof of the main theorem	114
3.6.1	Main intermediate results	114
3.6.2	End of the proof of the main theorem	115
3.6.3	Conclusions	116
3.7	The general covariant modulation procedure	125
3.7.1	Spacetime assumptions for the GCM procedure	125
3.7.2	Deformations of surfaces	128
3.7.3	Adapted frame transformations	128
3.7.4	GCM results	129
3.7.5	Main ideas	131
3.8	Overview of the proof of Theorems M0–M8	133
3.8.1	Discussion of Theorem M0	133
3.8.2	Discussion of Theorem M1	134
3.8.3	Discussion of Theorem M2	135
3.8.4	Discussion of Theorem M3	136
3.8.5	Discussion of Theorem M4	137
3.8.6	Discussion of Theorem M5	138
3.8.7	Discussion of Theorem M6	138
3.8.8	Discussion of Theorem M7	139
3.8.9	Discussion of Theorem M8	140

3.9	Structure of the rest of the book	143
4	Consequences of the Bootstrap Assumptions	145
4.1	Proof of Theorem M0	145
4.2	Control of averages and of the Hawking mass	164
4.2.1	Proof of Lemma 3.15	164
4.2.2	Proof of Lemma 3.16	172
4.3	Control of coordinates systems	174
4.4	Pointwise bounds for higher order derivatives	183
4.5	Proof of Proposition 3.20	188
4.6	Existence and control of the global frames	197
4.6.1	Proof of Proposition 3.23	197
4.6.2	Proof of Lemma 4.16	200
4.6.3	Proof of Proposition 3.26	208
5	Decay Estimates for q (Theorem M1)	213
5.1	Preliminaries	213
5.1.1	The foliation of \mathcal{M} by τ	214
5.1.2	Assumptions for Ricci coefficients and curvature	215
5.1.3	Structure of nonlinear terms	216
5.1.4	Main quantities	218
5.2	Proof of Theorem M1	223
5.2.1	Flux decay estimates for q	223
5.2.2	Proof of Theorem M1	224
5.2.3	Proof of Proposition 5.10	226
5.3	Improved weighted estimates	230
5.3.1	Basic and higher weighted estimates for wave equations . .	231
5.3.2	Proof of Theorem 5.14	233
5.3.3	Proof of Theorem 5.15	244
5.4	Decay estimates	249
5.4.1	First flux decay estimates	249
5.4.2	Flux decay estimates for \check{q}	253
5.4.3	Proof of Theorem 5.9	255
5.4.4	Proof of Proposition 5.12	259
5.4.5	Proof of Proposition 5.13	260
6	Decay Estimates for α and $\underline{\alpha}$ (Theorems M2, M3)	264
6.1	Proof of Theorem M2	264
6.1.1	A renormalized frame on $(^{ext})\mathcal{M}$	264
6.1.2	A transport equation for α	264
6.1.3	Estimates for transport equations in e_3	267
6.1.4	Decay estimates for α	271
6.1.5	End of the proof of Theorem M2	278
6.2	Proof of Theorem M3	279
6.2.1	Estimate for $\underline{\alpha}$ in $(^{int})\mathcal{M}$	279
6.2.2	Estimate for $\underline{\alpha}$ on Σ_*	281
6.2.3	Proof of Proposition 6.10	282
6.2.4	Proof of Lemma 6.12	286
6.2.5	Proof of Proposition 6.14	289
6.2.6	Proof of Lemma 6.16	292

7 Decay Estimates (Theorems M4, M5)	295
7.1 Preliminaries to the proof of Theorem M4	295
7.1.1 Geometric structure of Σ_*	295
7.1.2 Main assumptions	296
7.1.3 Basic lemmas	299
7.1.4 Main equations	301
7.1.5 Equations involving \mathbf{q}	302
7.1.6 Additional equations	305
7.2 Structure of the proof of Theorem M4	308
7.3 Decay estimates on the last slice Σ_*	311
7.3.1 Preliminaries	311
7.3.2 Differential identities involving GCM conditions on Σ_*	314
7.3.3 Control of the flux of some quantities on Σ_*	315
7.3.4 Estimates for some $\ell = 1$ modes on Σ_*	322
7.3.5 Decay of Ricci and curvature components on Σ_*	332
7.4 Control in $(^{ext})\mathcal{M}$, Part I	336
7.4.1 Preliminaries	336
7.4.2 Proposition 7.33	338
7.4.3 Estimates for $\check{\kappa}, \check{\mu}$ in $(^{ext})\mathcal{M}$	339
7.4.4 Estimates for the $\ell = 1$ modes in $(^{ext})\mathcal{M}$	340
7.4.5 Completion of the proof of Proposition 7.33	343
7.5 Control in $(^{ext})\mathcal{M}$, Part II	346
7.5.1 Estimate for η	347
7.5.2 Crucial lemmas	347
7.5.3 Proof of Proposition 7.35, Part I	355
7.5.4 Proof of Proposition 7.35, Part II	359
7.6 Conclusion of the proof of Theorem M4	362
7.7 Proof of Theorem M5	366
8 Initialization and Extension (Theorems M6, M7, M8)	372
8.1 Proof of Theorem M6	372
8.2 Proof of Theorem M7	376
8.3 Proof of Theorem M8	387
8.3.1 Main norms	389
8.3.2 Control of the global frame	391
8.3.3 Iterative procedure	393
8.3.4 End of the proof of Theorem M8	396
8.4 Proof of Proposition 8.7	399
8.4.1 A wave equation for $\tilde{\rho}$	399
8.4.2 Control of $\square_g(r)$	400
8.4.3 End of the proof of Proposition 8.7	405
8.5 Proof of Proposition 8.8	408
8.5.1 A wave equation for $\alpha + \Upsilon^2 \underline{\alpha}$	408
8.5.2 End of the proof of Proposition 8.8	417
8.6 Proof of Proposition 8.9	418
8.6.1 Control of α and $\Upsilon^2 \underline{\alpha}$	418
8.6.2 Control of $\underline{\alpha}$	420
8.6.3 End of the proof of Proposition 8.9	424
8.7 Proof of Proposition 8.10	424
8.7.1 r -weighted divergence identities for Bianchi pairs	425

8.7.2	End of the proof of Proposition 8.10	435
8.7.3	Proof of (8.3.12)	440
8.8	Proof of Proposition 8.11	442
8.8.1	Proof of Proposition 8.31	444
8.8.2	Weighted estimates for transport equations along e_4 in $(\text{ext})\mathcal{M}$	454
8.8.3	Several identities	460
8.8.4	Proof of Proposition 8.32	464
8.8.5	Proof of Proposition 8.33	471
8.9	Proof of Proposition 8.12	479
8.9.1	Weighted estimates for transport equations along e_3 in $(\text{int})\mathcal{M}$	480
8.9.2	Proof of Proposition 8.42	482
8.10	Proof of Proposition 8.13	485
9	GCM Procedure	486
9.1	Preliminaries	486
9.1.1	Main assumptions	488
9.1.2	Elliptic Hodge lemma	489
9.2	Deformations of S surfaces	489
9.2.1	Deformations	489
9.2.2	Pullback map	490
9.2.3	Comparison of norms between deformations	492
9.2.4	Adapted frame transformations	496
9.3	Frame transformations	504
9.3.1	Main GCM equations	513
9.3.2	Equation for the average of a	518
9.3.3	Transversality conditions	519
9.4	Existence of GCM spheres	520
9.4.1	The linearized GCM system	524
9.4.2	Comparison of the Hawking mass	526
9.4.3	Iteration procedure for Theorem 9.32	527
9.4.4	Existence and boundedness of the iterates	530
9.4.5	Convergence of the iterates	535
9.5	Proof of Proposition 9.37 and of Corollary 9.38	538
9.5.1	Proof of Proposition 9.37	538
9.5.2	Proof of Corollary 9.38	542
9.6	Proof of Proposition 9.43	545
9.6.1	Pullback of the main equations	545
9.6.2	Basic lemmas	548
9.6.3	Proof of the estimates (9.6.5), (9.6.6), (9.6.7)	556
9.7	A corollary to Theorem 9.32	559
9.8	Construction of GCM hypersurfaces	566
9.8.1	Definition of Σ_0	569
9.8.2	Extrinsic properties of Σ_0	570
9.8.3	Construction of Σ_0	583
10	Regge-Wheeler Type Equations	600
10.1	Basic Morawetz estimates	600
10.1.1	Structure of the proof of Theorem 10.1	601
10.1.2	A simplified set of assumptions	602
10.1.3	Functions depending on m and r	602

10.1.4 Deformation tensors of the vectorfields R, T, X	603
10.1.5 Basic integral identities	607
10.1.6 Main Morawetz identity	609
10.1.7 A first estimate	613
10.1.8 Improved lower bound in $(\text{ext})\mathcal{M}$	618
10.1.9 Cut-off correction in $(\text{int})\mathcal{M}$	625
10.1.10 The redshift vectorfield	632
10.1.11 Combined estimate	636
10.1.12 Lower bounds for \mathcal{Q}	642
10.1.13 First Morawetz estimate	644
10.1.14 Analysis of the error term \mathcal{E}_ϵ	651
10.1.15 Proof of Theorem 10.1	653
10.2 Dafermos-Rodnianski r^p -weighted estimates	656
10.2.1 Vectorfield $X = f(r)e_4$	659
10.2.2 Energy densities for $X = f(r)e_4$	659
10.2.3 Proof of Theorem 10.37	668
10.3 Higher weighted estimates	675
10.3.1 Wave equation for $\check{\psi}$	675
10.3.2 The r^p -weighted estimates for $\check{\psi}$	676
10.4 Higher derivative estimates	682
10.4.1 Basic assumptions	682
10.4.2 Strategy for recovering higher order derivatives	682
10.4.3 Commutation formulas with the wave equation	683
10.4.4 Some weighted estimates for wave equations	696
10.4.5 Proof of Theorem 5.17	701
10.4.6 Proof of Theorem 5.18	706
10.5 More weighted estimates for wave equations	711
A Appendix to Chapter 2	719
A.1 Proof of Proposition 2.64	719
A.2 Proof of Proposition 2.71	721
A.3 Proof of Lemma 2.72	725
A.4 Proof of Proposition 2.73	728
A.5 Proof of Proposition 2.74	733
A.6 Proof of Proposition 2.90	737
A.7 Proof of Lemma 2.92	750
A.8 Proof of Corollary 2.93	753
A.9 Proof of Lemma 2.91	755
A.10 Proof of Proposition 2.99	757
A.11 Proof of Proposition 2.100	760
A.12 Proof of the Teukolsky-Starobinsky identity	765
A.13 Proof of Proposition 2.107	773
A.14 Proof of Theorem 2.108	776
A.14.1 The Teukolsky equation for α	779
A.14.2 Commutation lemmas	781
A.14.3 Main commutation	788
A.14.4 Proof of Theorem 2.108	796
B Appendix to Chapter 8	799
B.1 Proof of Proposition 8.14	799

C Appendix to Chapter 9	806
C.1 Proof of Lemma 9.11	806
D Appendix to Chapter 10	819
D.1 Horizontal S -tensors	819
D.1.1 Mixed tensors	820
D.1.2 Invariant Lagrangian	820
D.1.3 Comparison of the Lagrangians	821
D.1.4 Energy-momentum tensor	822
D.2 Standard calculation	823
D.3 Vectorfield X_f	824
D.4 Proof of Proposition 10.47	827
Bibliography	836