

Contents

1	Introduction	1
1.1	Equivariant de Rham Poincaré Duality	1
1.2	Equivariant de Rham Gysin Morphisms	2
1.3	Adjunction Properties of Gysin Morphisms	4
1.4	Equivariant Cohomology Viewed as a Relative Cohomology Theory.....	5
1.5	Equivariant Poincaré Duality over Arbitrary Fields.....	6
1.6	Conditions on the Group G	7
1.7	Conditions on Topological Spaces	8
2	Nonequivariant Background	9
2.1	Category of Cochain Complexes.....	9
2.1.1	Fields in Use	9
2.1.2	Vector Spaces and Pairings	9
2.1.3	The Category $GV(\mathbb{k})$ of Graded Spaces	10
2.1.4	The Subcategories of Bounded Graded Spaces	10
2.1.5	Graded Algebras over Fields	10
2.1.6	The Category $DGV(\mathbb{k})$ of Differential Graded Vector Spaces	12
2.1.7	The Shift Functors	13
2.1.8	The Functors $\mathbf{Hom}_{\mathbb{k}}^{\bullet}(_, _)$ and $(_ \otimes_{\mathbb{k}} _)^{\bullet}$	14
2.1.9	On the Koszul Sign Rule	15
2.1.10	The Functor $\mathbf{Hom}_{\mathbb{k}}^{\bullet}(_, \mathbf{W})$	16
2.1.11	The Duality Functor	16
2.2	Categories of Manifolds.....	18
2.2.1	Manifolds.....	18
2.2.2	The Category of Manifolds.....	18
2.2.3	The Category of Manifolds and Proper Maps	18
2.2.4	G -Manifolds	19

2.3	Orientation and Integration	19
2.3.1	Orientability	19
2.3.2	Integration	21
2.4	Poincaré Duality	23
2.4.1	Poincaré Pairing	23
2.4.2	The Fundamental Class of an Oriented Manifold	29
2.5	Poincaré Adjunctions	30
2.5.1	Poincaré Adjoint Pairs	30
2.5.2	Manifolds and Maps of Finite de Rham Type	33
2.5.3	Ascending Chain Property	33
2.5.4	Existence of Proper Invariant Functions	34
2.5.5	Manifolds with Boundary	35
2.5.6	Proof of Proposition 2.5.3.1	36
2.6	The Gysin Functor	37
2.6.1	The Right Poincaré Adjunction Map	37
2.6.2	The Gysin Morphism	38
2.6.3	The Image of D'_M	42
2.7	The Gysin Functor for Proper Maps	44
2.8	Constructions of Gysin Morphisms	46
2.8.1	The Proper Case	46
2.8.2	The General Case	47
3	Poincaré Duality Relative to a Base Space	49
3.1	Fiber Bundles	49
3.1.1	Terminology	49
3.1.2	The Categories Top_B and Man_B	50
3.1.3	The Relative Point of View	51
3.1.4	Fiber Product	51
3.1.5	The Base Change Functor	54
3.1.6	Fiber Products of Fiber Bundles of Manifolds	56
3.1.7	Orientable Fiber Bundles	59
3.1.8	The Categories Man_B , Fib_B and Fib_B^{or}	62
3.1.9	Proper Subspaces of a Fiber Bundle	62
3.1.10	Differential Forms with Proper Supports	65
3.2	Integration Along Fibers on Fiber Bundles	71
3.2.1	The Case of Trivial Euclidean Bundles	72
3.2.2	Sheafification of Integration Along Fibers	78
3.2.3	Thom Class of an Oriented Vector Bundle	81
3.3	Poincaré Duality for Fiber Bundles	83
3.3.1	Sheafification of the Poincaré Adjunction	83
3.3.2	Deriving the Sheafified Poincaré Adjunction Functors	84
3.3.3	The Poincaré Duality Theorem for Fiber Bundles	85
3.3.4	Poincaré Duality for Fiber Bundles and Base Change	87

3.4	Poincaré Duality Relative to a Formal Base Space	90
3.4.1	Formality of Topological Spaces	91
3.4.2	Poincaré Duality Relative to Classifying Spaces	93
3.5	Gysin Morphisms for Fiber Bundles.....	95
3.5.1	Gysin Morphisms Relative to a Base Space	95
3.5.2	Gysin Morphisms for Fiber Bundles and Base Change	96
3.6	Examples of Gysin Morphisms	98
3.6.1	Adjointness of Gysin Morphism	98
3.6.2	Constant Map and Locally Trivial Fibrations.....	99
3.6.3	Open Embedding.....	100
3.6.4	Proper Base Change	100
3.6.5	Zero Section of a Vector Bundle	101
3.6.6	Closed Embedding	103
3.7	Applications.....	104
3.7.1	Gysin Long Exact Sequence.....	104
3.7.2	Lefschetz Fixed Point Theorem	106
3.8	Conclusion	107
4	Equivariant Background	109
4.1	Significant Dates in Equivariant Cohomology Theory.....	109
4.1.1	Cartan's ENS Seminar (1950)	109
4.1.2	Borel's IAS Seminar (1960)	114
4.1.3	Atiyah-Segal: Equivariant K -Theory (1968)	115
4.1.4	Quillen: Equivariant Cohomology (1971)	115
4.1.5	Hsiang's Book (1975).....	116
4.1.6	Atiyah-Bott and Berline-Vergne: Equivariant de Rham Cohomology (1980)	117
4.2	Category of \mathfrak{g} -Differential Graded Modules	119
4.2.1	Field in Use	119
4.2.2	The Category of \mathfrak{g} -Modules	119
4.2.3	\mathfrak{g} -Differential Graded Modules	120
4.2.4	\mathfrak{g} -Differential Graded Algebras	121
4.2.5	Split \mathfrak{g} -Complexes	122
4.3	Equivariant Cohomology of \mathfrak{g} -Complexes	124
4.3.1	The \mathfrak{g} -dg-Algebra $S(\mathfrak{g}^\vee)$	124
4.3.2	Cartan Complexes.....	124
4.3.3	Induced Morphisms on Cartan Complexes	126
4.3.4	Split G -Complexes.....	128
4.4	Equivariant Differential Forms.....	129
4.4.1	Fields in Use	129
4.4.2	G -Fundamental Vector Fields	129
4.4.3	Interior Products and Lie Derivatives	129
4.4.4	Complexes of Equivariant Differential Forms	130
4.4.5	On the Connectedness of G	131
4.4.6	Splitness of Complexes of Equivariant Differential Forms	134

4.5	Cohomological Properties of Homotopy Quotients.....	139
4.5.1	Local Triviality of G -Spaces	139
4.5.2	Slices	140
4.5.3	Existence of Slices	141
4.6	Constructing Classifying Spaces	146
4.6.1	The Milnor Construction	146
4.6.2	Stiefel Manifolds.....	147
4.6.3	Convention	148
4.7	The Borel Construction.....	148
4.7.1	The Homotopy Quotient Functor.....	148
4.7.2	On the Cohomology of the Homotopy Quotient	149
4.7.3	Orientability of the Homotopy Quotient	156
4.8	Equivariant de Rham Comparison Theorems	157
4.8.1	Question Iso 1	158
4.8.2	Question Iso 2	162
4.8.3	Equivariant Cohomology Comparison Theorem	163
4.9	Cohomology of Classifying Spaces.....	165
4.9.1	Canonicity of the Cohomology of Classifying Spaces	165
4.9.2	Formality of Classifying Spaces	166
4.10	Local Equivariant Cohomology	169
4.10.1	The Long Exact Sequence of Local Equivariant Cohomology.....	172
5	Equivariant Poincaré Duality.....	175
5.1	Differential Graded Modules over a Graded Algebra.....	175
5.1.1	Graded Modules and Algebras over Graded Algebras	175
5.1.2	The Category of Ω_G -Graded Modules	176
5.2	The Category of Ω_G -Differential Graded Modules	179
5.2.1	Definition	179
5.2.2	The $\mathbf{Hom}_{\Omega_G}^{\bullet}(_, _)$ and $(_ \otimes_{\Omega_G} _)^{\bullet}$ Bifunctors on $\mathbf{DGM}(\Omega_G)$	179
5.2.3	The Duality Functor on $\mathbf{DGM}(\Omega_G)$	180
5.2.4	The Forgetful Functor.....	180
5.2.5	On the Exactness of $\mathbf{Hom}^{\bullet}(_, _)$ and $(_ \otimes _)^{\bullet}$	181
5.3	Comparing the Categories $\mathcal{C}(\mathbf{GM}(\Omega_G))$ and $\mathbf{DGM}(\Omega_G)$	182
5.3.1	The \mathbf{Tot} Functors	182
5.3.2	The $\mathbf{Hom}_{\Omega_G}^{\bullet}(_, _)$ bifunctor on $\mathcal{C}(\mathbf{GM}(\Omega_G))$	184
5.3.3	The $(_ \otimes_{\Omega_G} _)^{\bullet}$ Bifunctor on $\mathcal{C}(\mathbf{GM}(\Omega_G))$	186
5.4	Deriving Functors in $\mathbf{GM}(\Omega_G)$	187
5.4.1	Augmentations	188
5.4.2	Simple Complex Associated with a Bicomplex	188
5.4.3	The $\mathbf{IR} \mathbf{Hom}_{\Omega_G}^{\bullet}(_, _)$ and $(_) \otimes_{\Omega_G}^{\mathbf{IL}} (_)$ Bifunctors on $\mathbf{GM}(\Omega_G)$	189
5.4.4	The \mathbf{Ext}^{\bullet} and \mathbf{Tor}^{\bullet} Bifunctors	191
5.4.5	The Duality Functor on $\mathcal{D}(\mathbf{GM}(\Omega_G))$	192

5.4.6	The Duality Functor on $\mathcal{D}(\text{DGM}(\Omega_G))$	192
5.4.7	Spectral Sequences Associated with $IR \text{ Hom}_{\Omega_G}^*(_, \Omega_G)$	193
5.5	Equivariant Integration	197
5.5.1	Definition	197
5.5.2	Equivariant Integration vs. Integration Along Fibers.....	199
5.6	Equivariant Poincaré Duality.....	201
5.6.1	The Ω_G -Poincaré Pairing	201
5.6.2	G -Equivariant Poincaré Duality Theorem	203
5.6.3	Torsion-Freeness, Freeness and Reflexivity	208
5.6.4	T -Equivariant Poincaré Duality Theorem	209
6	Equivariant Gysin Morphism and Euler Classes	211
6.1	G -Equivariant Gysin Morphism	211
6.1.1	Equivariant Finite de Rham Type Coverings.....	211
6.1.2	G -Equivariant Gysin Morphism for General Maps	212
6.1.3	G -Equivariant Gysin Morphism for Proper Maps.....	214
6.1.4	Gysin Morphisms through Spectral Sequences.....	215
6.2	Group Restriction and Equivariant Gysin Morphisms	216
6.2.1	Group Restriction and Equivariant Cohomology	216
6.2.2	Group Restriction and Integration.....	217
6.3	Adjointness of Equivariant Gysin Morphisms	218
6.3.1	Adjointness Property.....	218
6.4	Explicit Constructions of Equivariant Gysin Morphisms	220
6.4.1	Equivariant Open Embedding	220
6.4.2	Equivariant Constant Map	220
6.4.3	Equivariant Projection	221
6.4.4	Equivariant Fiber Bundle	222
6.4.5	Zero Section of an Equivariant Vector Bundle	222
6.4.6	Equivariant Gysin Long Exact Sequence	226
6.5	Equivariant Euler Classes	228
6.5.1	The Nonequivariant Euler Class.....	228
6.5.2	G -Equivariant Euler Class.....	229
6.5.3	G -Equivariant Euler Class of Fixed Points	230
6.5.4	T -Equivariant Euler Class of Fixed Points	230
7	Localization	235
7.1	The Localization Functor.....	235
7.2	Localized Equivariant Poincaré Duality	236
7.3	Localized Equivariant Gysin Morphisms.....	236
7.4	Torsion in Equivariant Cohomology Modules	237
7.4.1	Torsion.....	237
7.5	Slices and Orbit Types.....	238
7.5.1	The General Slice Theorem	238
7.5.2	Orbit Type of T -Manifolds	239
7.6	Localized Gysin Morphisms	240

7.7	The Localization Formula	242
7.7.1	Inversibility of Euler Classes	242
8	Changing the Coefficients Field	245
8.1	Comments about Notations	245
8.1.1	Preliminaries	246
8.2	Sheafification of Cartan Models over Arbitrary Fields	247
8.2.1	Dictionary	250
8.2.2	Reformulation of The Poincaré Adjunctions	251
8.3	Equivariant Poincaré Duality over Arbitrary Fields.....	256
8.3.1	The Equivariant Duality Theorem.....	256
8.4	Formality of IBG over Arbitrary Fields	257
8.4.1	The Integral Cohomology of G/T	257
8.5	Equivariant Gysin Morphisms over Arbitrary Fields	261
8.5.1	Gysin Morphism for General Maps	262
8.5.2	Gysin Morphism for Proper Maps	263
8.6	The Localization Formula over Arbitrary Fields	263
A	Basics on Derived Categories	265
A.1	Categories of Complexes	265
A.1.1	The Category of Complexes of an Abelian Category	266
A.1.2	Extending Additive Functors from Ab to $\mathcal{C}(\text{Ab})$	267
A.1.3	The Mapping Cone.....	267
A.1.4	Homotopies	268
A.1.5	The Homotopy Category $\mathcal{K}(\text{Ab})$	270
A.1.6	The Derived Category $\mathcal{D}(\text{Ab})$	275
A.1.7	The Subcategories $\mathcal{C}^*(\text{Ab})$, $\mathcal{K}^*(\text{Ab})$ and $\mathcal{D}^*(\text{Ab})$	279
A.2	Deriving Functors.....	279
A.2.1	Extending Functors from Ab to $\mathcal{D}(\text{Ab})$	279
A.2.2	Extending Functors from $\mathcal{C}(\text{Ab})$ to $\mathcal{K}(\text{Ab})$	279
A.2.3	Extending Functors from $\mathcal{K}(\text{Ab})$ to $\mathcal{D}(\text{Ab})$	280
A.2.4	Acyclic Resolutions.....	285
A.2.5	The Duality Functor on $\mathcal{D}(\text{DGM}(\Omega_G))$	287
A.3	DG-Modules over DG-Algebras	292
A.3.1	K -Injective (\mathcal{A}, d) -Differential Graded Modules	292
A.3.2	Formality of DGA's	293
A.3.3	Formality of DGM's	295
B	Sheaves of Differential Graded Algebras	301
B.1	Mild Topological Spaces	301
B.2	The Sheaf of Functions \mathcal{O}_X	302
B.3	Global Lifting of Germs on \mathcal{O}_X -modules.....	303
B.4	\mathcal{O}_X -Graded Algebras	305

- B.5 Localization Functor for \mathcal{O}_X -GA's 306
 - B.5.1 The Isomorphism $\mathbf{Hom}_{\mathcal{A}}^{\bullet}(\mathcal{A}, _)\simeq \Gamma(Y; _)$ 306
 - B.5.2 Right Adjoint to $\mathbf{Hom}_{\mathcal{A}}^{\bullet}(\mathcal{A}, _)$ 307
 - B.5.3 The Localization Functor for \mathcal{O}_X -GA's 308
- B.6 Equivalences of Some Derived Functors in $\mathcal{DGM}(\mathcal{A})$ 310
 - B.6.1 An Equivalence of Categories 310
 - B.6.2 Family of Supports 311
 - B.6.3 The functor Γ_{Φ} in $\mathbf{GM}(\mathcal{A}(X))$ 312
- B.7 \mathcal{O}_X -Differential Graded Algebras 315
 - B.7.1 Localization Functor for \mathcal{O}_X -DGA's 315
- B.8 The Localization Functor for \mathcal{O}_X -DGA's 316
- B.9 Equivalences of Derived Functors in $\mathcal{DDGM}(\mathcal{A}, d)$ 317
 - B.9.1 K -Injective Differential Graded Modules 317
- C Cartan's Theorem for \mathfrak{g} -dg-Ideals 323**
- D Graded Ring of Fractions 327**
- E Hints and Solutions to Exercises 329**
- References 359**
- Glossary 363**
- Index 369**