

# Contents

<b>1 Homotopy Theory and Arithmetic Geometry—Motivic and Diophantine Aspects: An Introduction</b> .....	1
Frank Neumann and Ambrus Pál	
1.1 Overview of Themes .....	1
1.2 Summaries of Individual Contributions .....	4
References .....	6
<b>2 An Introduction to <math>\mathbb{A}^1</math>-Enumerative Geometry</b> .....	11
Thomas Brazelton	
2.1 Introduction .....	11
2.2 Preliminaries .....	12
2.2.1 Enriching the Topological Degree .....	12
2.2.2 The Grothendieck–Witt Ring .....	14
2.2.3 Lannes’ Formula .....	16
2.2.4 The Unstable Motivic Homotopy Category .....	18
2.2.5 Colimits .....	21
2.2.6 Purity .....	24
2.3 $\mathbb{A}^1$ -enumerative Geometry .....	27
2.3.1 The Eisenbud–Khimshiashvili–Levine Signature Formula ...	28
2.3.2 Sketch of Proof for Theorem 4 .....	30
2.3.3 $\mathbb{A}^1$ -Milnor Numbers .....	32
2.3.4 An Arithmetic Count of the Lines on a Smooth Cubic Surface .....	35
2.3.5 An Arithmetic Count of the Lines Meeting 4 Lines in Space .....	40
References .....	44

<b>3</b>	<b>Cohomological Methods in Intersection Theory</b> .....	49
	Denis-Charles Cisinski	
3.1	Introduction .....	49
3.2	Étale Motives .....	51
3.2.1	The $h$ -topology .....	51
3.2.2	Construction of Motives, After Voevodsky .....	52
3.2.3	Functoriality .....	56
3.2.4	Representability Theorems .....	61
3.3	Finiteness and Euler Characteristic .....	63
3.3.1	Locally Constructible Motives .....	63
3.3.2	Integrality of Traces and Rationality of $\zeta$ -Functions.....	67
3.3.3	Grothendieck-Verdier Duality .....	75
3.3.4	Generic Base Change: A Motivic Variation on Deligne's Proof .....	79
3.4	Characteristic Classes .....	84
3.4.1	Künneth Formula .....	84
3.4.2	Grothendieck-Lefschetz Formula .....	89
	References .....	103
<b>4</b>	<b>Étale Homotopy and Obstructions to Rational Points</b> .....	107
	Tomer M. Schlank	
4.1	Introduction .....	107
4.2	$\infty$ -Categories.....	110
4.2.1	Motivation.....	110
4.2.2	Quasi-Categories.....	111
4.2.3	$\infty$ -Groupoids and the Homotopy Hypothesis .....	114
4.2.4	Quasi-Categories from Topological Categories .....	116
4.2.5	$\infty$ -Category Theory .....	117
4.2.6	The Homotopy Category .....	118
4.2.7	$\infty$ -Categories and Homological Algebra.....	119
4.2.8	Stable $\infty$ -Categories.....	120
4.2.9	Localization .....	121
4.3	$\infty$ -Topoi .....	122
4.3.1	Definitions .....	122
4.3.2	The Shape of an $\infty$ -Topos.....	124
4.4	Obstruction Theory .....	126
4.4.1	Obstruction Theory for Homotopy Types .....	127
4.4.2	For $\infty$ -Topoi and Linear(ized) Versions .....	131
4.5	Étale Homotopy and Rational Points.....	133
4.5.1	The étale $\infty$ -Topos .....	133
4.5.2	Rational Points.....	134
4.5.3	The Local-to-Global Principle .....	137
4.6	Galois Theory and Embedding Problems .....	140
4.6.1	Topoi and Embedding Problems .....	141
	References .....	143

<b>5</b>	<b><math>\mathbb{A}^1</math>-homotopy Theory and Contractible Varieties: A Survey</b> .....	145
	Aravind Asok and Paul Arne Østvær	
5.1	Introduction: Topological and Algebro-Geometric Motivations .....	145
5.1.1	Open Contractible Manifolds .....	146
5.1.2	Contractible Algebraic Varieties .....	149
5.2	A User’s Guide to $\mathbb{A}^1$ -homotopy Theory .....	153
5.2.1	Brief Topological Motivation .....	154
5.2.2	Homotopy Functors in Algebraic Geometry .....	155
5.2.3	The Unstable $\mathbb{A}^1$ -homotopy Category: Construction .....	157
5.2.4	The Unstable $\mathbb{A}^1$ -homotopy Category: Basic Properties.....	160
5.2.5	A Snapshot of the Stable Motivic Homotopy Category.....	166
5.3	Concrete $\mathbb{A}^1$ -weak Equivalences .....	169
5.3.1	Constructing $\mathbb{A}^1$ -weak Equivalences of Smooth Schemes ....	169
5.3.2	$\mathbb{A}^1$ -weak Equivalences vs. Weak Equivalences .....	171
5.3.3	Cancellation Questions and $\mathbb{A}^1$ -weak Equivalences .....	173
5.3.4	Danielewski Surfaces and Generalizations .....	174
5.3.5	Building Quasi-Affine $\mathbb{A}^1$ -contractible Varieties .....	176
5.4	Further Computations in $\mathbb{A}^1$ -homotopy Theory .....	179
5.4.1	$\mathbb{A}^1$ -homotopy Sheaves .....	180
5.4.2	$\mathbb{A}^1$ -connectedness and Geometry .....	182
5.4.3	$\mathbb{A}^1$ -homotopy Sheaves Spheres and Brouwer Degree.....	185
5.4.4	$\mathbb{A}^1$ -homotopy at Infinity .....	186
5.5	Cancellation Questions and $\mathbb{A}^1$ -contractibility .....	189
5.5.1	The Biregular Cancellation Problem .....	189
5.5.2	$\mathbb{A}^1$ -contractibility vs Topological Contractibility .....	190
5.5.3	Cancellation Problems and the Russell Cubic.....	193
5.5.4	$\mathbb{A}^1$ -contractibility of the Koras–Russell Threefold.....	202
5.5.5	Koras–Russell Fiber Bundles .....	204
	References .....	207
<b>Index</b> .....		<b>213</b>