

Contents

Part 0 Prelude

0	Finite Graphs	3
0.1	Graphs, Laplacians and Dirichlet forms	5
0.2	Characterizing forms associated to graphs	16
0.3	Characterizing Laplacians associated to graphs	22
0.4	Networks and electrostatics	28
0.5	The heat equation and the Markov property	40
0.6	Resolvents and heat semigroups	46
0.7	A Perron–Frobenius theorem and large time behavior	49
0.8	When there is no killing	57
0.9	Turning graphs into other graphs*	62
0.10	Markov processes and the Feynman–Kac formula*	67
	Exercises	82
	Notes	92

Part 1 Foundations and Fundamental Topics

1	Infinite Graphs – Key Concepts	97
1.1	The setting in a nutshell	97
1.2	Graphs and (regular) Dirichlet forms	109
1.3	Approximation, domain monotonicity and the Markov property	115
1.4	Connectedness, irreducibility and positivity improving operators ...	127
1.5	Boundedness and compactly supported functions	130
1.6	Graphs with standard weights	133
	Exercises	136
	Notes	139

2	Infinite Graphs – Toolbox	141
2.1	Generators, semigroups and resolvents on ℓ^p	142
2.2	Forms associated to graphs and restrictions to subsets	159
2.3	The curse of non-locality: Leibniz and chain rules	165
2.4	Creatures from the abyss*	169
2.5	Markov processes and the Feynman–Kac formula redux*	173
	Exercises	177
	Notes	184
3	Markov Uniqueness and Essential Self-Adjointness	185
3.1	Uniqueness of associated forms	185
3.2	Essential self-adjointness	189
3.3	Markov uniqueness	193
	Exercises	211
	Notes	214
4	Agmon–Allegretto–Piepenbrink and Persson Theorems	215
4.1	A local Harnack inequality and consequences	216
4.2	The ground state transform	222
4.3	The bottom of the spectrum	228
4.4	The bottom of the essential spectrum	230
	Exercises	237
	Notes	239
5	Large Time Behavior of the Heat Kernel	241
5.1	Positivity improving semigroups and the ground state	241
5.2	Theorems of Chavel–Karp and Li	245
5.3	The Neumann Laplacian and finite measure	249
	Exercises	252
	Notes	254
6	Recurrence	255
6.1	General preliminaries	261
6.2	The form perspective	268
6.3	The superharmonic function perspective	275
6.4	The Green’s function perspective	281
6.5	The Green’s formula perspective	290
6.6	A probabilistic point of view*	293
	Exercises	300
	Notes	305

7 Stochastic Completeness 307

7.1 The heat equation on ℓ^∞ 312

7.2 Stochastic completeness at infinity 318

7.3 The heat equation perspective 329

7.4 The Poisson equation perspective 333

7.5 The form perspective 337

7.6 The Green’s formula perspective 342

7.7 The Omori–Yau maximum principle 345

7.8 A stability criterion and Khasminskii’s criterion 348

7.9 A probabilistic interpretation* 352

Exercises 357

Notes 360

Part 2 Classes of Graphs

8 Uniformly Positive Measure 363

8.1 A Liouville theorem 364

8.2 Uniqueness of the form and essential self-adjointness 366

8.3 A spectral inclusion 368

8.4 The heat equation on ℓ^p 371

8.5 Graphs with standard weights 372

Exercises 375

Notes 377

9 Weak Spherical Symmetry 379

9.1 Symmetry of the heat kernel 383

9.2 The spectral gap 394

9.3 Recurrence 402

9.4 Stochastic completeness at infinity 405

Exercises 411

Notes 415

10 Sparseness and Isoperimetric Inequalities 417

10.1 Notions of sparseness 417

10.2 Co-area formulae 421

10.3 Weak sparseness and the form domain 423

10.4 Approximate sparseness and first order eigenvalue asymptotics 428

10.5 Sparseness and second order eigenvalue asymptotics 429

10.6 Isoperimetric inequalities and Weyl asymptotics 432

Exercises 436

Notes 439

Part 3 Geometry and Intrinsic Metrics

11	Intrinsic Metrics: Definition and Basic Facts	443
	11.1 Definition and motivation	443
	11.2 Path metrics and a Hopf–Rinow theorem	447
	11.3 Examples and relations to other metrics	453
	11.4 Geometric assumptions and cutoff functions	458
	Exercises	463
	Notes	466
12	Harmonic Functions and Caccioppoli Theory	469
	12.1 Caccioppoli inequalities	470
	12.2 Liouville theorems	483
	12.3 Applications of the Liouville theorems	489
	12.4 Shnol’ theorems	493
	Exercises	500
	Notes	504
13	Spectral Bounds	507
	13.1 Cheeger constants and lower spectral bounds	507
	13.2 Volume growth and upper spectral bounds	513
	Exercises	523
	Notes	524
14	Volume Growth Criterion for Stochastic Completeness and Uniqueness Class	525
	14.1 Uniqueness class	526
	14.2 Refinements	538
	14.3 Volume growth criterion for stochastic completeness	545
	Exercises	548
	Notes	550

Appendix

A	The Spectral Theorem	553
B	Closed Forms on Hilbert spaces	587
C	Dirichlet Forms and Beurling–Deny Criteria	599
D	Semigroups, Resolvents and their Generators	605

E Aspects of Operator Theory	623
E.1 A characterization of the resolvent	623
E.2 The discrete and essential spectrum	626
E.3 Reducing subspaces and commuting operators	638
E.4 The Riesz–Thorin interpolation theorem	644
References	647
Index	663
Notation Index	667