Contents

1 Introduction **1**

1.1 The Einstein-Klein-Gordon Coupled System 1

1.1.1 Wave Coordinates and PDE Formulation of the Problem 2

1.2 The Global Regularity Theorem 5

1.2.1 Global Stability Results in General Relativity 8

1.2.2 Simplified Wave-Klein-Gordon Models 9

1.2.3 Small Data Global Regularity Results 10

1.2.4 Assumptions on the Initial Data 12

1.2.5 The Mini-bosons 12

1.3 Main Ideas and Further Asymptotic Results 13

1.3.1 Global Nonlinear Stability 13

1.3.2 Nonlinear Scattering 16

1.3.3 Asymptotic Bounds and Causal Geodesics 17

1.3.4 Weak Peeling Estimates 19

1.3.5 The ADM Energy and the Linear Momentum 20

1.3.6 The Bondi Energy 21

1.3.7 Organization 22

1.3.8 Acknowledgements 23

2 The Main Construction and Outline of the Proof **24**

2.1 Setup and the Main Bootstrap Proposition 24

2.1.1 The Nonlinearities $\mathcal{N}^h_{\alpha\beta}$ and $\mathcal{N}^\psi$ 24

2.1.2 The Fourier Transform and Frequency Projections 27

2.1.3 Vector-fields 28

2.1.4 Decomposition of the Metric Tensor 28

2.1.5 Linear Profiles and the Z-norms 30

2.1.6 The Main Bootstrap Proposition 32

2.2 Outline of the Proof 34

2.2.1 Chapter 3: Preliminary Estimates 34

2.2.2 Chapter 4: the Nonlinearities $\mathcal{N}^h_{\alpha\beta}$ and $\mathcal{N}^\psi$ 37

2.2.3 Chapter 5: Improved Energy Estimates 41

2.2.4 Chapter 6: Improved Profile Bounds 44

3 Preliminary Estimates **48**

3.1 Some Lemmas 48

3.1.1 General Lemmas 48
3.1.2 The Phases $\Phi_{\sigma\mu\nu}$ 50
3.1.3 Elements of Paradifferential Calculus 59
3.2 Linear and Bilinear Estimates 62
3.2.1 Linear Estimates 62
3.2.2 Multipliers and Bilinear Operators 70
3.2.3 Bilinear Estimates 71
3.2.4 Interpolation Inequalities 77
3.3 Analysis of the Linear Profiles 79

4 The Nonlinearities $\mathcal{N}^h_{\alpha\beta}$ and $\mathcal{N}^\psi$ 87
4.1 Localized Bilinear Estimates 87
4.1.1 Frequency Localized L^2 Estimates 87
4.1.2 The Classes of Functions $\mathcal{G}_a$ 102
4.2 Bounds on the Nonlinearities $\mathcal{N}^h_{\alpha\beta}$ and $\mathcal{N}^\psi$ 108
4.2.1 The Quadratic Nonlinearities 109
4.2.2 The Cubic and Higher Order Nonlinearities 118
4.2.3 Additional Low Frequency Bounds 121
4.2.4 Additional Bounds on Some Quadratic Nonlinearities . . 122
4.3 Decompositions of the Main Nonlinearities 125
4.3.1 The Variables $F^{\mathcal{L}}, \underline{F}^{\mathcal{L}}, \rho^{\mathcal{L}}, \omega_j^{\mathcal{L}}, \Omega_j^{\mathcal{L}}, \vartheta_{jk}^{\mathcal{L}}$ 125
4.3.2 Energy Disposable Nonlinearities 127
4.3.3 Null Structures 133
4.3.4 The Main Decomposition 135

5 Improved Energy Estimates 147
5.1 Setup and Preliminary Reductions 147
5.1.1 Energy Increments 147
5.1.2 The Main Spacetime Bounds 149
5.1.3 Poincaré Normal Forms 153
5.1.4 Paralinearization of the Reduced Wave Operator 154
5.2 Pure Wave Interactions 158
5.2.1 Null Interactions 158
5.2.2 Non-null Semilinear Terms 168
5.2.3 Second Symmetrization and Paradifferential Calculus . . 174
5.3 Mixed Wave-Klein-Gordon Interactions 185

6 Improved Profile Bounds 197
6.1 Weighted Bounds 197
6.2 Z-norm Control of the Klein-Gordon Field 198
6.2.1 Renormalization 199
6.2.2 Improved Control 200
6.3 Z-norm Control of the Metric Components 213
6.3.1 The First Reduction 215
6.3.2 The Nonlinear Terms $\mathcal{R}_a^{h_{\alpha\beta}}$, $a \in \{1, 2, 4, 6\}$ 216

6.3.3 Localized Bilinear Wave Interactions 222
6.3.4 The Nonlinear Terms $\mathcal{R}_3^{h_{\alpha\beta}}$ 234
6.3.5 The Nonlinear Terms $\mathcal{R}_5^{h_{\alpha\beta}}$ 237

7 The Main Theorems **241**
7.1 Global Regularity and Asymptotics 241
7.1.1 Global Regularity 241
7.1.2 Decay of the Metric and the Klein-Gordon Field 244
7.1.3 Null and Timelike Geodesics 254
7.2 Weak Peeling Estimates and the ADM Energy 258
7.2.1 Peeling Estimates 258
7.2.2 The ADM Energy 264
7.2.3 The Linear Momentum 270
7.2.4 Gauge Conditions and Parameterizations 272
7.3 Asymptotically Optical Functions and the Bondi Energy 275
7.3.1 Almost Optical Functions and the Friedlander Fields . . . 275
7.3.2 The Bondi Energy 283
7.3.3 The interior energy 288

Bibliography **291**

Index **297**