

Paolo Toni • Pier Domenico Lamberti  
Giacomo Drago

# 100+1 Problems in Advanced Calculus

A Creative Journey through the Fjords  
of Mathematical Analysis for Beginners

 Springer

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