

# Contents

Preface .....	v
<b>Chapter 1 Motivation. One-Dimensional Plasticity and Viscoplasticity ...</b>	<b>1</b>
1.1 Overview .....	1
1.2 Motivation. One-Dimensional Frictional Models .....	2
1.2.1 Local Governing Equations.....	2
1.2.2 An Elementary Model for (Isotropic) Hardening Plasticity .....	9
1.2.3 Alternative Form of the Loading/Unloading Conditions .....	13
1.2.4 Further Refinements of the Hardening Law .....	17
1.2.5 Geometric Properties of the Elastic Domain .....	19
1.3 The Initial Boundary-Value Problem .....	21
1.3.1 The Local Form of the IBVP .....	22
1.3.2 The Weak Formulation of the IBVP .....	24
1.3.3 Dissipation. A priori Stability Estimate .....	26
1.3.4 Uniqueness of the Solution to the IBVP. Contractivity .....	29
1.3.5 Outline of the Numerical Solution of the IBVP .....	31
1.4 Integration Algorithms for Rate-Independent Plasticity .....	32
1.4.1 The Incremental Form of Rate-Independent Plasticity .....	33
1.4.2 Return-Mapping Algorithms. Isotropic Hardening .....	35
1.4.3 Discrete Variational Formulation. Convex Optimization .....	39
1.4.4 Extension to the Combined Isotropic/Kinematic Hardening Model .....	43
1.5 Finite-Element Solution of the Elastoplastic IBVP. An Illustration ..	46
1.5.1 Spatial Discretization. Finite-Element Approximation .....	46
1.5.2 Incremental Solution Procedure .....	49
1.6 Stability Analysis of the Algorithmic IBVP .....	53
1.6.1 Algorithmic Approximation to the Dynamic Weak Form .....	54
1.7 One-Dimensional Viscoplasticity .....	57
1.7.1 One-Dimensional Rheological Model .....	58
1.7.2 Dissipation. A Priori Stability Estimate .....	64
1.7.3 An Integration Algorithm for Viscoplasticity .....	66

<b>Chapter 2 Classical Rate-Independent Plasticity and Viscoplasticity . . . . .</b>	<b>71</b>
2.1 Review of Some Standard Notation . . . . .	72
2.1.1 The Local Form of the IBVP. Elasticity . . . . .	73
2.2 Classical Rate-Independent Plasticity . . . . .	75
2.2.1 Strain-Space and Stress-Space Formulations . . . . .	75
2.2.2 Stress-Space Governing Equations . . . . .	76
2.2.3 Strain-Space Formulation . . . . .	82
2.2.4 An Elementary Example: 1-D Plasticity . . . . .	85
2.3 Plane Strain and 3-D, Classical $J_2$ Flow Theory . . . . .	89
2.3.1 Perfect Plasticity . . . . .	89
2.3.2 $J_2$ Flow Theory with Isotropic/Kinematic Hardening . . . . .	90
2.4 Plane-Stress $J_2$ Flow Theory . . . . .	91
2.4.1 Projection onto the Plane-Stress Subspace . . . . .	92
2.4.2 Constrained Plane-Stress Equations . . . . .	92
2.5 General Quadratic Model of Classical Plasticity . . . . .	95
2.5.1 The Yield Criterion . . . . .	96
2.5.2 Evolution Equations. Elastoplastic Moduli . . . . .	96
2.6 The Principle of Maximum Plastic Dissipation . . . . .	98
2.6.1 Classical Formulation. Perfect Plasticity . . . . .	98
2.6.2 General Associative Hardening Plasticity in Stress Space . . . . .	101
2.6.3 Interpretation of Associative Plasticity as a Variational Inequality . . . . .	103
2.7 Classical (Rate-Dependent) Viscoplasticity . . . . .	105
2.7.1 Formulation of the Basic Governing Equations . . . . .	105
2.7.2 Interpretation as a Viscoplastic Regularization . . . . .	105
2.7.3 Penalty Formulation of the Principle of Maximum Plastic Dissipation . . . . .	106
2.7.4 The Generalized Duvaut-Lions Model . . . . .	110
<b>Chapter 3 Integration Algorithms for Plasticity and Viscoplasticity . . . . .</b>	<b>113</b>
3.1 Basic Algorithmic Setup. Strain-Driven Problem . . . . .	114
3.1.1 Associative plasticity . . . . .	115
3.2 The Notion of Closest Point Projection . . . . .	115
3.2.1 Plastic Loading. Discrete Kuhn–Tucker Conditions . . . . .	116
3.2.2 Geometric Interpretation . . . . .	118
3.3 Example 3.1. $J_2$ Plasticity. Nonlinear Isotropic/Kinematic Hardening . . . . .	120
3.3.1 Radial Return Mapping . . . . .	120
3.3.2 Exact Linearization of the Algorithm . . . . .	122
3.4 Example 3.2. Plane-Stress $J_2$ Plasticity. Kinematic/Isotropic Hardening . . . . .	125
3.4.1 Return-Mapping Algorithm . . . . .	126
3.4.2 Consistent Elastoplastic Tangent Moduli . . . . .	127
3.4.3 Implementation . . . . .	128

3.4.4 Accuracy Assessment. Isoerror Maps.....	131
3.4.5 Closed-Form Exact Solution of the Consistency Equation .....	133
3.5 Interpretation. Operator Splits and Product Formulas .....	139
3.5.1 Example 3.3. Lie's Formula.....	139
3.5.2 Elastic-Plastic Operator Split.....	140
3.5.3 Elastic Predictor. Trial Elastic State .....	140
3.5.4 Plastic Corrector. Return Mapping .....	141
3.6 General Return-Mapping Algorithms .....	143
3.6.1 General Closest Point Projection.....	143
3.6.2 Consistent Elastoplastic Moduli. Perfect Plasticity .....	145
3.6.3 Cutting-Plane Algorithm.....	148
3.7 Extension of General Algorithms to Viscoplasticity .....	149
3.7.1 Motivation. $J_2$ -Viscoplasticity .....	150
3.7.2 Closest Point Projection .....	151
3.7.3 A Note on Notational Conventions .....	151
<b>Chapter 4 Discrete Variational Formulation and Finite-Element Implementation .....</b>	<b>154</b>
4.1 Review of Some Basic Notation .....	155
4.1.1 Gateaux Variation .....	156
4.1.2 The Functional Derivative.....	158
4.1.3 Euler–Lagrange Equations .....	159
4.2 General Variational Framework for Elastoplasticity .....	161
4.2.1 Variational Characterization of Plastic Response.....	162
4.2.2 Discrete Lagrangian for elastoplasticity.....	163
4.2.3 Variational Form of the Governing Equations .....	165
4.2.4 Extension to Viscoplasticity .....	167
4.3 Finite-Element Formulation. Assumed-Strain Method .....	168
4.3.1 Matrix and Vector Notation .....	168
4.3.2 Summary of Governing Equations .....	170
4.3.3 Discontinuous Strain and Stress Interpolations .....	170
4.3.4 Reduced Residual. Generalized Displacement Model .....	171
4.3.5 Closest Point Projection Algorithm .....	172
4.3.6 Linearization. Consistent Tangent Operator .....	174
4.3.7 Matrix Expressions .....	176
4.3.8 Variational Consistency of Assumed-Strain Methods .....	176
4.4 Application. B-Bar Method for Incompressibility .....	178
4.4.1 Assumed-Strain and Stress Fields .....	178
4.4.2 Weak Forms .....	179
4.4.3 Discontinuous Volume/Mean-Stress Interpolations .....	180
4.4.4 Implementation 1. B-Bar-Approach .....	181
4.4.5 Implementation 2. Mixed Approach .....	182
4.4.6 Examples and Remarks on Convergence .....	183
4.5 Numerical Simulations .....	183
4.5.1 Plane-Strain $J_2$ Flow Theory .....	184

4.5.2 Plane-Stress $J_2$ Flow Theory . . . . .	188
<b>Chapter 5 Nonsmooth Multisurface Plasticity and Viscoplasticity . . . . .</b>	<b>198</b>
5.1 Rate-Independent Multisurface Plasticity. Continuum Formulation . . . . .	199
5.1.1 Summary of Governing Equations . . . . .	199
5.1.2 Loading/Unloading Conditions . . . . .	201
5.1.3 Consistency Condition. Elastoplastic Tangent Moduli . . . . .	204
5.1.4 Geometric Interpretation . . . . .	204
5.2 Discrete Formulation. Rate-Independent Elastoplasticity . . . . .	206
5.2.1 Closest Point Projection Algorithm for Multisurface Plasticity . . . . .	206
5.2.2 Loading/Unloading. Discrete Kuhn–Tucker Conditions . . . . .	208
5.2.3 Solution Algorithm and Implementation . . . . .	209
5.2.4 Linearization: Algorithmic Tangent Moduli . . . . .	212
5.3 Extension to Viscoplasticity . . . . .	215
5.3.1 Motivation. Perzyna-Type Models . . . . .	216
5.3.2 Extension of the Duvaut–Lions Model . . . . .	217
5.3.3 Discrete Formulation . . . . .	217
<b>Chapter 6 Numerical Analysis of General Return Mapping Algorithms . . . . .</b>	<b>219</b>
6.1 Motivation: Nonlinear Heat Conduction . . . . .	220
6.1.1 The Continuum Problem . . . . .	221
6.1.2 The Algorithmic Problem . . . . .	225
6.1.3 Nonlinear Stability Analysis . . . . .	226
6.2 Infinitesimal Elastoplasticity . . . . .	228
6.2.1 The Continuum Problem for Plasticity and Viscoplasticity . . . . .	228
6.2.2 The Algorithmic Problem . . . . .	235
6.2.3 Nonlinear Stability Analysis . . . . .	237
6.3 Concluding Remarks . . . . .	239
<b>Chapter 7 Nonlinear Continuum Mechanics and Phenomenological Plasticity Models . . . . .</b>	<b>240</b>
7.1 Review of Some Basic Results in Continuum Mechanics . . . . .	240
7.1.1 Configurations. Basic Kinematics . . . . .	241
7.1.2 Motions. Lagrangian and Eulerian Descriptions . . . . .	245
7.1.3 Rate of Deformation Tensors . . . . .	248
7.1.4 Stress Tensors. Equations of Motion . . . . .	250
7.1.5 Objectivity. Elastic Constitutive Equations . . . . .	252
7.1.6 The Notion of Isotropy. Isotropic Elastic Response . . . . .	259
7.2 Variational Formulation. Weak Form of Momentum Balance . . . . .	262
7.2.1 Configuration Space and Admissible Variations . . . . .	262
7.2.2 The Weak Form of Momentum Balance . . . . .	264
7.2.3 The Rate Form of the Weak Form of Momentum Balance . . . . .	266

<b>7.3 Ad Hoc Extensions of Phenomenological Plasticity Based on Hypoelastic Relationships .....</b>	269
7.3.1 Formulation in the Spatial Description.....	269
7.3.2 Formulation in the Rotated Description.....	271
<b>Chapter 8 Objective Integration Algorithms for Rate Formulations of Elastoplasticity.....</b>	<b>276</b>
8.1 Objective Time-Stepping Algorithms.....	278
8.1.1 The Geometric Setup .....	279
8.1.2 Approximation for the Rate of Deformation Tensor .....	281
8.1.3 Approximation for the Lie Derivative .....	283
8.1.4 Application: Numerical Integration of Rate Constitutive Equations .....	285
8.2 Application to $J_2$ Flow Theory at Finite Strains .....	287
8.2.1 A $J_2$ Flow Theory .....	288
8.3 Objective Algorithms Based on the Notion of a Rotated Configuration .....	290
8.3.1 Objective Integration of Elastoplastic Models .....	291
8.3.2 Time-Stepping Algorithms for the Orthogonal Group .....	295
<b>Chapter 9 Phenomenological Plasticity Models Based on the Notion of an Intermediate Stress-Free Configuration .....</b>	<b>300</b>
9.1 Kinematic Preliminaries. The (Local) Intermediate Configuration ..	301
9.1.1 Micromechanical Motivation. Single-Crystal Plasticity .....	301
9.1.2 Kinematic Relationships Associated with the Intermediate Configuration .....	302
9.1.3 Deviatoric-Volumetric Multiplicative Split .....	305
9.2 $J_2$ Flow Theory at Finite Strains. A Model Problem .....	306
9.2.1 Formulation of the Governing Equations .....	307
9.3 Integration Algorithm for $J_2$ Flow Theory .....	311
9.3.1 Integration of the Flow Rule and Hardening Law .....	311
9.3.2 The Return-Mapping Algorithm .....	314
9.3.3 Exact Linearization of the Algorithm .....	320
9.4 Assessment of the Theory. Numerical Simulations .....	322
<b>Chapter 10 Viscoelasticity.....</b>	<b>336</b>
10.1 Motivation. One-Dimensional Rheological Models .....	337
10.1.1 Formulation of the Constitutive Model .....	338
10.1.2 Convolution Representation .....	339
10.1.3 Generalized Relaxation Models .....	343
10.2 Three-Dimensional Models: Formulation Restricted to Linearized Kinematics .....	347
10.2.1 Formulation of the Model .....	347
10.2.2 Thermodynamic Aspects. Dissipation .....	349
10.3 Integration Algorithms .....	351

10.3.1 Algorithmic Internal Variables and Finite-Element Database.....	351
10.3.2 One-Step, Unconditionally Stable and Second-Order Accurate Recurrence Formula.....	353
10.3.3 Linearization, Consistent Tangent Moduli .....	355
10.4 Finite Elasticity with Uncoupled Volume Response .....	358
10.4.1 Volumetric/Deviatoric Multiplicative Split.....	358
10.4.2 Stored-Energy Function and Stress Response .....	359
10.4.3 Elastic Tangent Moduli .....	361
10.5 A Class of Nonlinear, Viscoelastic, Constitutive Models .....	364
10.5.1 Formulation of the Nonlinear Viscoelastic Constitutive Model .....	364
10.6 Implementation of Integration Algorithms for Nonlinear Viscoelasticity.....	367
10.6.1 One-Step, Second-Order Accurate Recurrence Formula. ....	367
10.6.2 Configuration Update Procedure.....	369
10.6.3 Consistent (Algorithmic) Tangent Moduli .....	370
<b>References .....</b>	<b>375</b>
<b>Index .....</b>	<b>389</b>