Contents

Preface xi

Introduction xiii

Acknowledgments xxi

Chapter 0. Basic Notation and Background 1
- 0.1. Sets and set notation 1
- 0.2. The basic numbers systems 6
- 0.3. Logical Notation and Relations 7
- 0.4. Mathematical induction 9
- 0.5. Functions 10
- 0.6. Orbit Structure 12
- 0.7. Counting sets: combinatorics 13
- 0.8. Geometric Series 15
- 0.9. Exponentials 15

Chapter 1. Division with Remainder, Place Value, and Order of Magnitude 17
- 1.1. Division with Remainder (DwR) 18
- 1.2. The Euclidean Algorithm 20
- 1.3. Base-10 place value representation 24
- 1.4. Base-b representations of real numbers 25
- 1.5. Order of magnitude and significant digits 27
- 1.6. Exercises 29

Chapter 2. Modular congruence and base-b expansions of rational numbers 33
- 2.1. Modular congruence 34
- 2.2. Modular congruence of arithmetic expressions 35
- 2.3. Modular congruence and base-b: A fast track to $r_m(a)$ 37
- 2.4. The base-b representation of a rational number 40
- 2.5. Exercises 46

Chapter 3. Discrete Real Additive Groups 49
- 3.1. Metric Spaces 50
- 3.2. Additive groups and translation-invariant metrics 53
- 3.3. Discrete real additive groups 58
- 3.4. Discrete subgroups of $\mathbb{R}^n$ 58
- 3.5. Commensurability, gcd, and lcm 59
- 3.6. Some basic properties of gcd and lcm 61
- 3.7. The group $\mathbb{R}^\bullet$ and its discrete subgroups 65
- 3.8. Appendix: Frobenius numbers 67

3.9. Appendix: Compactness and connectedness 70
3.10. Exercises 74

Chapter 4. Primes and Factorization 79
4.1. Primes and prime factorization 79
4.2. Prime power factorization 81
4.3. Prime-valued polynomials 83
4.4. Cryptography 84
4.5. Exercises 89

Chapter 5. Group Theory and Euclidean Isometry Groups 93
5.1. Binary operations 94
5.2. Semigroups, groups, and subgroups 96
5.3. Homomorphisms, isomorphisms, automorphisms 97
5.4. G-sets 99
5.5. Quotient groups G/N 102
5.6. Torsion: elements of finite order 103
5.7. Finite cyclic groups 105
5.8. Isometries of $\mathbb{R}^n$ $(n \leq 3)$ 107
5.9. The Euclidean isometry groups $\mathbb{E}(n) = \text{Isom}(\mathbb{R}^n)$ 109
5.10. Appendix: Platonic solids and finite isometry groups of $\mathbb{R}^3$ 113
5.11. Appendix: p-groups and Sylow Theorems 120
5.12. Exercises 122

Chapter 6. Rings and Fields 127
6.1. Rings and fields 128
6.2. Quotient rings 131
6.3. Zero-divisors, nilpotents, idempotents, units, and primes 132
6.4. Chinese remainder theorem 134
6.5. The multiplicative group $(\mathbb{Z}/\mathbb{Z}m)^\bullet$ 135
6.6. Quadratic Reciprocity 138
6.7. Polynomial Rings 142
6.8. Unique Factorization 145
6.9. Unique Factorization Domains (UFDs) 150
6.10. Partial Fractions 153
6.11. Appendix: The ABC Conjecture 156
6.16. Exercises 171

Chapter 7. Polynomials 175
7.1. Polynomial division with remainder (PDwR) and Roots 176
7.2. Polynomial interpolation 179
7.3. Binomial Theorem and Multinomial Theorem 180
7.4. The Inclusion-Exclusion Formula (I-EF) 183
7.5. Derangements 185
7.6. Symmetric functions 186
7.7. Multi-linearity and symmetry 188
7.8. Integer polynomial dynamics 195
7.9. Complex polynomial dynamics 197
7.10. The landscape of polynomial equations 204
7.11. Exercises 207

Chapter 8. Combinatorics 211
8.1. What is combinatorics? 212
8.2. Binomial coefficients and the set of injections $[d] \to [n]$ 212
8.3. The Frobenius Endomorphism and Binomial congruences 216
8.4. Graphs and trees. $V - E + F = 2$ 218
8.5. Multinomial coefficients and surjections $[n] \to [r]$ 220
8.6. Partitions 223
8.7. Mobius inversion 228
8.8. Sperner's theorem 234
8.9. Binary sequences and Catalan numbers $C(n)$ 235
8.10. Some of the many incarnations of $C(n)$ 238
8.11. Discrete Probability 243
8.12. Exercises 258

Chapter 9. Discrete Calculus 263
9.1. Sequences as functions $f : \mathbb{N} \to \mathbb{R}$; $\Delta, S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ 264
9.2. The Fundamental Theorem of Discrete Calculus 266
9.3. The Binomial Polynomials $B_d(x) = \binom{x}{d}$ 267
9.4. The Binomial Taylor Expansion 268
9.5. Binomial representations of x^d 270
9.6. Sums of d^{th} powers: $S_d(n+1) = 1^d + 2^d + \cdots + n^d$ 271
9.7. Geometric representations of $\mathcal{S}_d(n+1)$ 273
9.8. $\mathcal{S}_d$ as a polynomial in S_1 275
9.9. Exercises 277

Chapter 10. Complex numbers 279
10.1. Definition and models of $\mathbb{C}$ 280
10.2. Discrete additive subgroups and subrings of $\mathbb{C}$ 282
10.3. The complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$ 283
10.4. The multiplicative group structure of $\mathbb{C}$ 285
10.5. Roots of unity 286
10.6. Discrete subgroups of $\mathbb{C}^\bullet$ 287
10.7. The Fundamental Theorem of Algebra 288
10.8. $\mathbb{C}$ and Euclidean Transformations 290
10.9. Pythagorean Theorem, Pythagorean Triples, and Gaussian Integers 290
10.10. Quaternions 296
10.11. Exercises 304

Chapter 11. Mathematical Connections 307
11.1. Connection-oriented mathematical thinking 307
11.2. Connection-making in solving a problem 310
11.3. Connections between mathematical problems 316
11.4. Different problems reducible to a common model 322
11.5. Problems related to the Euclidean Algorithm 327
11.6. The magical marriage of two games 332

Index 335