

Contents

| | | |
|----------|---|----|
| 1 | Introduction | 1 |
| 1.1 | What Is a Cube Complex? | 2 |
| 1.2 | What Is the CAT(0) Property?..... | 3 |
| 1.3 | Which Cube Complexes Are CAT(0)?..... | 4 |
| 1.4 | Why Should One Study CAT(0) Cube Complexes?..... | 5 |
| 1.5 | Where to Look for Cube Complexes?..... | 7 |
| 1.6 | What's in This Book? | 7 |
| 1.7 | Some (of the Many) Things I Have Omitted..... | 9 |
| 1.8 | Final Remarks | 10 |
| 2 | Metric Spaces Meet Groups | 11 |
| 2.1 | A Tiny Bit of Metric Geometry | 11 |
| 2.2 | Groups as Metric Spaces..... | 18 |
| 2.3 | Exercises..... | 24 |
| 3 | Non-positive Curvature | 27 |
| 3.1 | Metric Spaces of Non-positive Curvature..... | 27 |
| 3.1.1 | Model Spaces of Constant Sectional Curvature | 28 |
| 3.1.2 | Definition of CAT(0) Spaces | 29 |
| 3.1.3 | Properties of CAT(0) Spaces | 31 |
| 3.2 | Angles in CAT(0) Spaces..... | 36 |
| 3.3 | Flat Cones | 39 |
| 3.4 | Exercises..... | 44 |
| 4 | Cube Complexes and Gromov's Link Condition | 47 |
| 4.1 | Polyhedral Complexes and the Link Condition..... | 47 |
| 4.1.1 | The Local Structure of Cell Complexes | 51 |
| 4.2 | CAT(0) Cube Complexes | 56 |
| 4.3 | Cube Completions | 67 |
| 4.4 | Right Angled Artin Groups..... | 70 |
| 4.5 | Exercises..... | 76 |

| | | |
|----------|---|-----|
| 5 | Hyperplanes and Half-Spaces | 79 |
| 5.1 | Hyperplanes | 79 |
| 5.2 | Half-Space Systems | 85 |
| 5.3 | Roller Duality | 99 |
| 5.4 | Exercises..... | 104 |
| 6 | Cubulating Coxeter Groups | 109 |
| 6.1 | Coxeter Groups: Definition and Examples..... | 109 |
| 6.2 | The Deletion Condition | 115 |
| 6.3 | Walls in Coxeter Groups..... | 117 |
| 6.4 | Half-Space System for Coxeter Groups..... | 122 |
| 6.5 | Exercises..... | 129 |
| 7 | A Panoramic Tour | 133 |
| 7.1 | Fixed Point Properties of Group Actions | 133 |
| 7.1.1 | Actions on Trees | 134 |
| 7.1.2 | Helly's Theorem for CAT(0) Cube Complexes | 138 |
| 7.2 | The Tits Alternative | 143 |
| 7.2.1 | HNN-Extensions and Free Amalgamations | 144 |
| 7.2.2 | Ends of Groups | 147 |
| 7.2.3 | Fuchsian Groups | 152 |
| 7.2.4 | The Tits Alternative..... | 152 |
| 7.3 | Special Cube Complexes | 155 |
| 7.3.1 | What Makes a Cube Complex Special? | 156 |
| 7.3.2 | Special Cubulated Groups Embed in RAAGs..... | 161 |
| 7.3.3 | Cube Complexes Meet 3-Dimensional Manifolds | 164 |
| 7.4 | Phylogenetic Trees | 165 |
| 7.4.1 | Modeling Mutation Using Trees | 166 |
| 7.4.2 | Exploring the Structure of n -Trees..... | 167 |
| 7.4.3 | The Space of Phylogenetic Trees | 169 |
| 7.5 | Exercises..... | 171 |
| | References | 173 |
| | Index | 179 |