

# Contents

<b>Foreword</b>	vii
<b>Preface</b>	xi
<b>Guide to the Main Mathematical Concepts and Their Application</b>	xix
<b>Notation and Symbols</b>	xxi
<b>1 Introduction</b>	1
1.1 The Image Society . . . . .	1
1.2 What Is a Digital Image? . . . . .	3
1.3 About Partial Differential Equations (PDEs) . . . . .	5
1.4 Detailed Plan . . . . .	5
<b>2 Mathematical Preliminaries</b>	23
How to Read This Chapter . . . . .	23
2.1 The Direct Method in the Calculus of Variations . . . . .	24
2.1.1 Topologies on Banach Spaces . . . . .	24
2.1.2 Convexity and Lower Semicontinuity . . . . .	26
2.1.3 Relaxation . . . . .	31
2.1.4 About $\Gamma$ -Convergence . . . . .	34
2.2 The Space of Functions of Bounded Variation . . . . .	36
2.2.1 Basic Definitions on Measures . . . . .	36

2.2.2	Definition of $BV(\Omega)$	39
2.2.3	Properties of $BV(\Omega)$	40
2.2.4	Convex Functions of Measures	44
2.3	Viscosity Solutions in PDEs	44
2.3.1	About the Eikonal Equation	44
2.3.2	Definition of Viscosity Solutions	46
2.3.3	About the Existence	48
2.3.4	About the Uniqueness	49
2.4	Elements of Differential Geometry: Curvature	51
2.4.1	Parametrized Curves	52
2.4.2	Curves as Isolevel of a Function $u$	52
2.4.3	Images as Surfaces	53
2.5	Other Classical Results Used in This Book	53
2.5.1	Inequalities	54
2.5.2	Calculus Facts	55
2.5.3	About Convolution and Smoothing	56
2.5.4	Uniform Convergence	57
2.5.5	Dominated Convergence Theorem	57
2.5.6	Well-Posed Problems	58
<b>3</b>	<b>Image Restoration</b>	<b>59</b>
How to Read This Chapter		59
3.1	Image Degradation	60
3.2	The Energy Method	62
3.2.1	An Inverse Problem	62
3.2.2	Regularization of the Problem	63
3.2.3	Existence and Uniqueness of a Solution for the Minimization Problem	66
3.2.4	Toward the Numerical Approximation	70
	A Quadratic Approximation	71
	The Half-Quadratic Minimization	72
3.2.5	Some Invariances and the Role of $\lambda$	77
3.2.6	Some Remarks on the Nonconvex Case	80
3.3	PDE-Based Methods	84
3.3.1	Smoothing PDEs	85
	The Heat Equation	85
	Nonlinear Diffusion	88
	The Alvarez–Guichard–Lions–Morel	
	Scale Space Theory	97
	Weickert’s Approach	103
	Surface Based Approaches	107
3.3.2	Smoothing–Enhancing PDEs	111
	The Perona and Malik Model	111
	Regularization of the Perona and Malik Model: Catté et al.	113

3.3.3	Enhancing PDEs . . . . .	118
	The Osher and Rudin Shock Filters . . . . .	118
	A Case Study: Construction of a Solution by the Method of Characteristics . . . . .	120
	Comments on the Shock-Filter Equation . . . . .	124
<b>4</b>	<b>The Segmentation Problem</b>	<b>129</b>
	How to Read This Chapter . . . . .	129
4.1	Definition and Objectives . . . . .	130
4.2	The Mumford and Shah Functional . . . . .	133
4.2.1	A Minimization Problem . . . . .	133
4.2.2	The Mathematical Framework for the Existence of a Solution . . . . .	133
4.2.3	Regularity of the Edge Set . . . . .	141
4.2.4	Approximations of the Mumford and Shah Functional . . . . .	146
4.2.5	Experimental Results . . . . .	151
4.3	Geodesic Active Contours and the Level-Set Method . . . . .	152
4.3.1	The Kass–Witkin–Terzopoulos model . . . . .	153
4.3.2	The Geodesic Active Contours Model . . . . .	155
4.3.3	The Level-Set Method . . . . .	162
4.3.4	Experimental Results . . . . .	174
4.3.5	About Some Recent Advances . . . . .	176
	Global Stopping Criterion . . . . .	176
	Toward More General Shape Representation . . . . .	178
<b>5</b>	<b>Other Challenging Applications</b>	<b>181</b>
	How to Read This Chapter . . . . .	181
5.1	Sequence Analysis . . . . .	182
5.1.1	Introduction . . . . .	182
5.1.2	The Optical Flow: An Apparent Motion . . . . .	184
	The Optical Flow Constraint (OFC) . . . . .	185
	Solving the Aperture Problem . . . . .	186
	Overview of a Discontinuity-Preserving Variational Approach . . . . .	190
	Alternatives to the OFC . . . . .	193
5.1.3	Sequence Segmentation . . . . .	195
	Introduction . . . . .	195
	A Variational Formulation . . . . .	196
	Mathematical Study of the Time-Sampled Energy . . . . .	199
	Experiments . . . . .	202
5.1.4	Sequence Restoration . . . . .	204
5.2	Image Classification . . . . .	210
5.2.1	Introduction . . . . .	210

5.2.2	A Level-Set Approach for Image Classification . . . . .	210
5.2.3	A Variational Model for Image Classification and Restoration . . . . .	218
<b>A</b>	<b>Introduction to Finite Difference Methods</b>	<b>229</b>
	How to Read This Chapter . . . . .	229
A.1	Definitions and Theoretical Considerations Illustrated by the 1-D Parabolic Heat Equation . . . . .	230
A.1.1	Getting Started . . . . .	230
A.1.2	Convergence . . . . .	233
A.1.3	The Lax Theorem . . . . .	235
A.1.4	Consistency . . . . .	235
A.1.5	Stability . . . . .	237
A.2	Hyperbolic Equations . . . . .	242
A.3	Difference Schemes in Image Analysis . . . . .	251
A.3.1	Getting Started . . . . .	251
A.3.2	Image Restoration by Energy Minimization . . . . .	255
A.3.3	Image Enhancement by the Osher and Rudin Shock Filters . . . . .	259
A.3.4	Curve Evolution with the Level-Set Method . . . . .	259
	Mean Curvature Motion . . . . .	261
	Constant Speed Evolution . . . . .	262
	The Pure Advection Equation . . . . .	263
	Image Segmentation by the Geodesic Active Contour Model . . . . .	264
<b>References</b>		<b>265</b>
<b>Index</b>		<b>283</b>