Contents

Preface IX

1 Inverse Problems and Interpretation of Measurements 1
1.1 Introductory Examples 3
1.2 Inverse Crimes 5

2 Classical Regularization Methods 7
2.1 Introduction: Fredholm Equation 7
2.2 Truncated Singular Value Decomposition 10
2.3 Tikhonov Regularization 16
2.3.1 Generalizations of the Tikhonov Regularization 24
2.4 Regularization by Truncated Iterative Methods 27
2.4.1 Landweber–Fridman Iteration 27
2.4.2 Kaczmarz Iteration and ART 31
2.4.3 Krylov Subspace Methods 39
2.5 Notes and Comments 46

3 Statistical Inversion Theory 49
3.1 Inverse Problems and Bayes' Formula 50
3.1.1 Estimators 52
3.2 Construction of the Likelihood Function 55
3.2.1 Additive Noise 56
3.2.2 Other Explicit Noise Models 58
3.2.3 Counting Process Data 60
3.3 Prior Models 62
3.3.1 Gaussian Priors 62
3.3.2 Impulse Prior Densities 62
3.3.3 Discontinuities 65
3.3.4 Markov Random Fields 66
3.3.5 Sample-based Densities 70
3.4 Gaussian Densities 72

3.4.1 Gaussian Smoothness Priors 79
3.5 Interpreting the Posterior Distribution 90
3.6 Markov Chain Monte Carlo Methods 91
3.6.1 The Basic Idea 91
3.6.2 Metropolis–Hastings Construction of the Kernel 94
3.6.3 Gibbs Sampler 98
3.6.4 Convergence 106
3.7 Hierarcical Models 108
3.8 Notes and Comments 112

4 Nonstationary Inverse Problems 115
4.1 Bayesian Filtering 115
4.1.1 A Nonstationary Inverse Problem 116
4.1.2 Evolution and Observation Models 118
4.2 Kalman Filters 123
4.2.1 Linear Gaussian Problems 123
4.2.2 Extended Kalman Filters 126
4.3 Particle Filters 129
4.4 Spatial Priors 133
4.5 Fixed-lag and Fixed-interval Smoothing 138
4.6 Higher-order Markov Models 140
4.7 Notes and Comments 143

5 Classical Methods Revisited 145
5.1 Estimation Theory 146
5.1.1 Maximum Likelihood Estimation 146
5.1.2 Estimators Induced by Bayes Costs 147
5.1.3 Estimation Error with Affine Estimators 149
5.2 Test Cases 150
5.2.1 Prior Distributions 150
5.2.2 Observation Operators 152
5.2.3 The Additive Noise Models 155
5.2.4 Test Problems 157
5.3 Sample-Based Error Analysis 158
5.4 Truncated Singular Value Decomposition 159
5.5 Conjugate Gradient Iteration 162
5.6 Tikhonov Regularization 164
5.6.1 Prior Structure and Regularization Level 166
5.6.2 Misspecification of the Gaussian Observation Error Model 170
5.6.3 Additive Cauchy Errors 173
5.7 Discretization and Prior Models 175
5.8 Statistical Model Reduction, Approximation Errors and Inverse Crimes 181
5.8.1 An Example: Full Angle Tomography and CGNE 184

5.9 Notes and Comments 186

6 Model Problems 189
6.1 X-ray Tomography 189
6.1.1 Radon Transform 190
6.1.2 Discrete Model 192
6.2 Inverse Source Problems 194
6.2.1 Quasi-static Maxwell's Equations 194
6.2.2 Electric Inverse Source Problems 197
6.2.3 Magnetic Inverse Source Problems 198
6.3 Impedance Tomography 202
6.4 Optical Tomography 208
6.4.1 The Radiation Transfer Equation 208
6.4.2 Diffusion Approximation 211
6.4.3 Time-harmonic Measurement 219
6.5 Notes and Comments 219

7 Case Studies 223
7.1 Image Deblurring and Recovery of Anomalies 223
7.1.1 The Model Problem 223
7.1.2 Reduced and Approximation Error Models 225
7.1.3 Sampling the Posterior Distribution 229
7.1.4 Effects of Modelling Errors 234
7.2 Limited Angle Tomography: Dental X-ray Imaging 236
7.2.1 The Layer Estimation 239
7.2.2 MAP Estimates 240
7.2.3 Sampling: Gibbs Sampler 241
7.3 Biomagnetic Inverse Problem: Source Localization 242
7.3.1 Reconstruction with Gaussian White Noise Prior Model 243
7.3.2 Reconstruction of Dipole Strengths with the ℓ^1-prior Model 245
7.4 Dynamic MEG by Bayes Filtering 249
7.4.1 A Single Dipole Model 250
7.4.2 More Realistic Geometry 253
7.4.3 Multiple Dipole Models 254
7.5 Electrical Impedance Tomography: Optimal Current Patterns . 260
7.5.1 A Posteriori Synthesized Current Patterns 260
7.5.2 Optimization Criterion 262
7.5.3 Numerical Examples 265
7.6 Electrical Impedance Tomography: Handling Approximation Errors 269
7.6.1 Meshes and Projectors 270
7.6.2 The Prior Distribution and the Prior Model 272
7.6.3 The Enhanced Error Model 273

7.6.4 The MAP Estimates 275
7.7 Electrical Impedance Process Tomography 278
7.7.1 The Evolution Model 280
7.7.2 The Observation Model and the Computational Scheme 283
7.7.3 The Fixed-lag State Estimate 285
7.7.4 Estimation of the Flow Profile 286
7.8 Optical Tomography in Anisotropic Media 291
7.8.1 The Anisotropy Model 292
7.8.2 Linearized Model 296
7.9 Optical Tomography: Boundary Recovery 299
7.9.1 The General Elliptic Case 300
7.9.2 Application to Optical Diffusion Tomography 303
7.10 Notes and Comments 305

A Appendix: Linear Algebra and Functional Analysis 311
A.1 Linear Algebra 311
A.2 Functional Analysis 314
A.3 Sobolev Spaces 316

B Appendix 2: Basics on Probability 319
B.1 Basic Concepts 319
B.2 Conditional Probabilities 323

References 329

Index 337