Contents

Preface xi

Introduction. Painlevé Transcendents as Nonlinear Special Functions 1

Part 1. Riemann-Hilbert Problem, Isomonodromy Method and Special Functions 37

Chapter 1. Systems of Linear Ordinary Differential Equations with Rational Coefficients. Elements of the General Theory 39
0. Introduction. The basic notations and facts 39
1. The local theory 44
2. The global theory 59

Chapter 2. Monodromy Theory and Special Functions 65
0. Monodromy group 65
1. Two by two systems. Special functions 75

Chapter 3. Inverse Monodromy Problem and Riemann-Hilbert Factorization 97
0. Introduction 97
1. Riemann-Hilbert problems 97
2. The Fuchsian Systems. Hilbert's twenty-first problem. 106
3. Two by two systems 108

Chapter 4. Isomonodromy Deformations. The Painlevé Equations 133
0. General setting 133
1. The Fuchsian case. The Schlesinger equations 142
2. Isomonodromy deformations for two by two systems: The Painlevé equations 145

Chapter 5. The Isomonodromy Method 161
0. The second Painlevé transcendent 161
1. The fourth Painlevé transcendent 181
2. The first Painlevé transcendent 188
3. The third Painlevé transcendent 195
4. The fifth Painlevé transcendent 203
5. Global solutions of the Painlevé equations 211

Chapter 6. Bäcklund Transformations 219
0. Introduction 219
1. PII equation 222
2. PIII equation 224

3. PIV equation 226

Part 2. Asymptotics of the Painlevé II Transcendent. A Case Study 231

Chapter 7. Asymptotic Solutions of the Second Painlevé Equation in the Complex Plane. Direct Monodromy Problem Approach 233
0. Introduction 233
1. Preliminary remarks. The Boutroux ansatz 234
2. Direct monodromy problem. Formulation of the main theorem 237
3. WKB-analysis for the Ψ-function. 240
4. Local solutions near turning points 245
5. Approximation of the monodromy data 247
6. Uniformization by theta-functions. Justification of the Boutroux ansatz 252

Chapter 8. Asymptotic Solutions of the Second Painlevé Equation in the Complex Plane. Inverse Monodromy Problem Approach 261
0. Preliminary remarks. The Deift-Zhou method 261
1. RH parametrization of PII in the complex plane 265
2. The transformation of the RH problem 268
3. Construction of the function $g(z)$ 276
4. The model Baker-Akhiezer RH problem 278
5. Local RH problems near the branch points 283
6. Asymptotic solution of the main RH problem 290
7. Asymptotics of the Painlevé function 295

Chapter 9. PII Asymptotics on the Canonical Six-rays. The Purely Imaginary Case 297
0. Introduction 297
1. Formulation of results. Main theorems and discussion 298
2. The direct monodromy problem approach. The proof of Theorem 9.3 303
3. The direct monodromy problem approach. The proof of Theorem 9.4 313
4. The inverse monodromy problem approach. The proof of Theorem 9.1. 318
5. The inverse monodromy problem approach. The proof of Theorem 9.2 331

Chapter 10. PII Asymptotics on the Canonical Six-rays. Real-valued Case 349
0. Introduction 349
1. Formulation of results. Main theorems and discussion 350
2. Proof of Theorem 10.1: Asymptotics of the Painlevé functions as $x \to +\infty$ 353
3. Proof of Theorem 10.2: Asymptotics of the Painlevé functions as $x \to -\infty$ 368

Chapter 11. PII Quasi-linear Stokes Phenomenon 373
0. Introduction 373

1. Linear Stokes phenomenon revisited 374
2. General remarks on the Stokes phenomenon for PII 379
3. The RH problem for PII 381
4. Special points of the monodromy surface 387
5. Nonspecial points of the monodromy surface 394
6. Asymptotic solution for $s_2 = 0$ 412
7. The Hastings-McLeod solution 428

Part 3. Asymptotics of the Third Painlevé Transcendent 433

Chapter 12. PIII Equation, an Overview 435
0. Introduction. Generic transforms and elementary solutions 435
1. Rational solutions of PIII 437
2. Algebraic solutions of PIII 439

Chapter 13. Sine-Gordon Reduction of PIII 443
0. Introduction 443
1. Direct monodromy problem for the sine-Gordon PIII reduction 446
2. Inverse monodromy problem 452
3. Exponential solutions of SG-PIII via the inverse monodromy problem 454

Chapter 14. Canonical Four-rays. Real-valued Solutions of SG-PIII 459
0. Introduction 459
1. Monodromy data parametrization at $x = 0$. 460
2. Monodromy data parametrization at infinity 464
3. The asymptotics of nonsingular solutions 468

Chapter 15. Canonical Four-rays. Singular Solutions of the SG-PIII 479
0. Introduction 479
1. Monodromy data and the Laurent series expansion 481
2. Ψ-function asymptotics at a pole 484
3. Mathieu equation 486
4. Asymptotic distribution of poles 490

Chapter 16. Asymptotics in the Complex Plane of the SG-PIII Transcendent 495
0. Introduction 495
1. Direct monodromy problem 499
2. WKB approximation of the Ψ-function 500
3. Asymptotics at the turning points 504
4. Zeros of the elliptic function ansatz 508

Appendix A. Proof of Theorem 3.4 511

Appendix B. The Birkhoff-Grothendieck Theorem with a Parameter 531

Bibliography 539

Subject Index 551