

Contents

Preface	v
Notation and Terminology	xv
CHAPTER I	
Two-Dimensional Manifolds	1
§1. Introduction	1
§2. Definition and Examples of n -Manifolds	2
§3. Orientable vs. Nonorientable Manifolds	3
§4. Examples of Compact, Connected 2-Manifolds	5
§5. Statement of the Classification Theorem for Compact Surfaces	9
§6. Triangulations of Compact Surfaces	14
§7. Proof of Theorem 5.1	16
§8. The Euler Characteristic of a Surface	26
References	34
CHAPTER II	
The Fundamental Group	35
§1. Introduction	35
§2. Basic Notation and Terminology	36
§3. Definition of the Fundamental Group of a Space	38
§4. The Effect of a Continuous Mapping on the Fundamental Group	42
§5. The Fundamental Group of a Circle Is Infinite Cyclic	47

§6. Application: The Brouwer Fixed-Point Theorem in Dimension 2	50
§7. The Fundamental Group of a Product Space	52
§8. Homotopy Type and Homotopy Equivalence of Spaces	54
References	59

CHAPTER III

Free Groups and Free Products of Groups	60
§1. Introduction	60
§2. The Weak Product of Abelian Groups	60
§3. Free Abelian Groups	63
§4. Free Products of Groups	71
§5. Free Groups	75
§6. The Presentation of Groups by Generators and Relations	78
§7. Universal Mapping Problems	81
References	85

CHAPTER IV

Seifert and Van Kampen Theorem on the Fundamental Group of the Union of Two Spaces. Applications	86
§1. Introduction	86
§2. Statement and Proof of the Theorem of Seifert and Van Kampen	87
§3. First Application of Theorem 2.1	91
§4. Second Application of Theorem 2.1	95
§5. Structure of the Fundamental Group of a Compact Surface	96
§6. Application to Knot Theory	103
§7. Proof of Lemma 2.4	108
References	116

CHAPTER V

Covering Spaces	117
§1. Introduction	117
§2. Definition and Some Examples of Covering Spaces	117
§3. Lifting of Paths to a Covering Space	123
§4. The Fundamental Group of a Covering Space	126
§5. Lifting of Arbitrary Maps to a Covering Space	127
§6. Homomorphisms and Automorphisms of Covering Spaces	130

§7. The Action of the Group $\pi(X, x)$ on the Set $p^{-1}(x)$	133
§8. Regular Covering Spaces and Quotient Spaces	135
§9. Application: The Borsuk–Ulam Theorem for the 2-Sphere	138
§10. The Existence Theorem for Covering Spaces	140
References	146

CHAPTER VI

Background and Motivation for Homology Theory	147
§1. Introduction	147
§2. Summary of Some of the Basic Properties of Homology Theory	147
§3. Some Examples of Problems which Motivated the Development of Homology Theory in the Nineteenth Century	149
References	157

CHAPTER VII

Definitions and Basic Properties of Homology Theory	158
§1. Introduction	158
§2. Definition of Cubical Singular Homology Groups	158
§3. The Homomorphism Induced by a Continuous Map	163
§4. The Homotopy Property of the Induced Homomorphisms	166
§5. The Exact Homology Sequence of a Pair	169
§6. The Main Properties of Relative Homology Groups	173
§7. The Subdivision of Singular Cubes and the Proof of Theorem 6.4	178

CHAPTER VIII

Determination of the Homology Groups of Certain Spaces: Applications and Further Properties of Homology Theory	186
§1. Introduction	186
§2. Homology Groups of Cells and Spheres—Applications	192
§3. Homology of Finite Graphs	201
§4. Homology of Compact Surfaces	206
§5. The Mayer–Vietoris Exact Sequence	207
§6. The Jordan–Brouwer Separation Theorem and Invariance of Domain	211
§7. The Relation between the Fundamental Group and the First Homology Group	217
References	224

CHAPTER IX

Homology of CW-Complexes **225**

§1. Introduction	225
§2. Adjoining Cells to a Space	225
§3. CW-Complexes	228
§4. The Homology Groups of a CW-Complex	232
§5. Incidence Numbers and Orientations of Cells	238
§6. Regular CW-Complexes	243
§7. Determination of Incidence Numbers for a Regular Cell Complex	244
§8. Homology Groups of a Pseudomanifold	249
References	253

CHAPTER X

Homology with Arbitrary Coefficient Groups **254**

§1. Introduction	254
§2. Chain Complexes	254
§3. Definition and Basic Properties of Homology with Arbitrary Coefficients	262
§4. Intuitive Geometric Picture of a Cycle with Coefficients in G	266
§5. Coefficient Homomorphisms and Coefficient Exact Sequences	267
§6. The Universal Coefficient Theorem	269
§7. Further Properties of Homology with Arbitrary Coefficients	274
References	278

CHAPTER XI

The Homology of Product Spaces **279**

§1. Introduction	279
§2. The Product of CW-Complexes and the Tensor Product of Chain Complexes	280
§3. The Singular Chain Complex of a Product Space	282
§4. The Homology of the Tensor Product of Chain Complexes (The Künneth Theorem)	284
§5. Proof of the Eilenberg–Zilber Theorem	286
§6. Formulas for the Homology Groups of Product Spaces	300
References	303

CHAPTER XII

Cohomology Theory	305
§1. Introduction	305
§2. Definition of Cohomology Groups—Proofs of the Basic Properties	306
§3. Coefficient Homomorphisms and the Bockstein Operator in Cohomology	309
§4. The Universal Coefficient Theorem for Cohomology Groups	310
§5. Geometric Interpretation of Cochains, Cocycles, etc.	316
§6. Proof of the Excision Property; the Mayer–Vietoris Sequence	319
References	322

CHAPTER XIII

Products in Homology and Cohomology	323
§1. Introduction	323
§2. The Inner Product	324
§3. An Overall View of the Various Products	324
§4. Extension of the Definition of the Various Products to Relative Homology and Cohomology Groups	329
§5. Associativity, Commutativity, and Existence of a Unit of the Various Products	333
§6. Digression: The Exact Sequence of a Triple or a Triad	336
§7. Behavior of Products with Respect to the Boundary and Coboundary Operator of a Pair	338
§8. Relations Involving the Inner Product	341
§9. Cup and Cap Products in a Product Space	342
§10. Remarks on the Coefficients for the Various Products—The Cohomology Ring	343
§11. The Cohomology of Product Spaces (The Künneth Theorem for Cohomology)	344
References	349

CHAPTER XIV

Duality Theorems for the Homology of Manifolds	350
§1. Introduction	350
§2. Orientability and the Existence of Orientations for Manifolds	351
§3. Cohomology with Compact Supports	358
§4. Statement and Proof of the Poincaré Duality Theorem	360

§5. Applications of the Poincaré Duality Theorem to Compact Manifolds	365
§6. The Alexander Duality Theorem	370
§7. Duality Theorems for Manifolds with Boundary	375
§8. Appendix: Proof of Two Lemmas about Cap Products	380
References	393

CHAPTER XV

Cup Products in Projective Spaces and Applications of Cup Products **394**

§1. Introduction	394
§2. The Projective Spaces	394
§3. The Mapping Cylinder and Mapping Cone	399
§4. The Hopf Invariant	402
References	406

APPENDIX A

A Proof of De Rham's Theorem **407**

§1. Introduction	407
§2. Differentiable Singular Chains	408
§3. Statement and Proof of De Rham's Theorem	411
References	417

APPENDIX B

Permutation Groups or Transformation Groups **419**

§1. Basic Definitions	419
§2. Homogeneous G -spaces	421

Index	424
-------	-----