

# CONTENTS

Preface . . . . .	v
Reader's Guide . . . . .	ix
<b>I SPECTRAL THEORY OF SELF-ADJOINT OPERATORS</b>	
1 Domains, Adjoints, Resolvents and Spectra . . . . .	2
2 Resolutions of the Identity . . . . .	9
3 Representation Theorems . . . . .	15
4 The Spectral Theorem . . . . .	18
5 Quadratic Forms and Self-adjoint Operators . . . . .	23
6 Self-adjoint Extensions of Symmetric Operators . . . . .	29
7 Problems . . . . .	31
8 Notes and Complements . . . . .	41
<b>II SCHRÖDINGER OPERATORS</b>	
1 The Free Hamiltonians . . . . .	44
2 Schrödinger Operators as Perturbations . . . . .	49
2.1 Self-adjointness . . . . .	50
2.2 Perturbation of the Absolutely Continuous Spectrum	57
2.3 An Approximation Argument . . . . .	62
3 Path Integral Formulas . . . . .	62
3.1 Brownian Motions and the Free Hamiltonians . . . . .	63
3.2 The Feynman-Kac Formula . . . . .	68
4 Eigenfunctions . . . . .	73
4.1 $L^2$ -Eigenfunctions . . . . .	73
4.2 The Periodic Case . . . . .	77
4.3 Generalized Eigenfunction Expansions . . . . .	79
5 Problems . . . . .	82
6 Notes and Complements . . . . .	86
<b>III ONE-DIMENSIONAL SCHRÖDINGER OPERATORS</b>	
1 The Continuous Case . . . . .	90
1.1 Essential Self-adjointness . . . . .	90
1.2 The Operator in an Interval . . . . .	95
1.3 Green's and Weyl-Titchmarsh's Functions . . . . .	99
1.4 The Propagator . . . . .	112

1.5 Examples . . . . .	116
2 The Lattice Case . . . . .	121
3 Approximations of the Spectral Measures . . . . .	130
4 Spectral Types . . . . .	135
4.1 Absolutely Continuous Spectrum . . . . .	135
4.2 Singular Spectrum . . . . .	139
4.3 Pure Point Spectrum . . . . .	141
5 Quasi-one Dimensional Schrödinger Operators . . . . .	144
5.1 The Schrödinger Operator in a Strip . . . . .	144
5.2 Approximation of the Spectral Measures . . . . .	152
5.3 Nature of the Spectrum . . . . .	158
6 Problems . . . . .	164
7 Notes and Complements . . . . .	171
<b>IV PRODUCTS OF RANDOM MATRICES</b>	<b>175</b>
1 General Ergodic Theorems . . . . .	176
2 Matrix Valued Systems . . . . .	178
3 Group Action on Compact Spaces . . . . .	185
3.1 Definitions and Notations . . . . .	185
3.2 Laplace Operators on the Space of Continuous Functions . . . . .	189
3.3 The Laplace Operators on the Space of Hölder Continuous Functions . . . . .	194
4 Products of Independent Random Matrices . . . . .	196
4.1 The Upper Lyapunov Exponent . . . . .	197
4.2 The Lyapunov Spectrum . . . . .	200
4.3 Schrödinger Matrices . . . . .	209
5 Markovian Multiplicative Systems . . . . .	215
5.1 The Upper Lyapunov Exponent . . . . .	217
5.2 The Lyapunov Spectrum . . . . .	220
5.3 Laplace Transform . . . . .	223
6 Boundaries of the Symplectic Group . . . . .	227
7 Problems . . . . .	232
8 Notes and Comments . . . . .	238
<b>V ERGODIC FAMILIES OF SELF-ADJOINT OPERATORS</b>	<b>241</b>
1 Measurability Concepts . . . . .	242
2 Spectra of Ergodic Families . . . . .	247
3 The Case of Random Schrödinger Operators . . . . .	257
3.1 Examples . . . . .	261
4 Regularity Properties of the Lyapunov Exponents . . . . .	267

4.1	Subharmonicity . . . . .	269
4.2	Continuity . . . . .	272
4.3	Local Hölder Continuity . . . . .	275
4.4	Smoothness . . . . .	280
5	Problems . . . . .	289
6	Notes and Complements . . . . .	294
<b>VI THE INTEGRATED DENSITY OF STATES</b>		<b>299</b>
1	Existence Problems . . . . .	300
1.1	Setting of the Problem . . . . .	300
1.2	Path Integral Approach . . . . .	301
1.3	Functional Analytic Approach . . . . .	308
2	Asymptotic Behavior and Lifschitz Tails . . . . .	313
2.1	Tauberian Arguments . . . . .	314
2.2	The Anderson Model . . . . .	321
3	More on the Lattice Case . . . . .	326
4	The One Dimensional Cases . . . . .	333
4.1	The Continuous Case . . . . .	333
4.2	The Lattice Case . . . . .	338
5	Problems . . . . .	348
6	Notes and Complements . . . . .	353
<b>VII ABSOLUTELY CONTINUOUS SPECTRUM AND INVERSE THEORY</b>		<b>359</b>
1	The $w$ -function . . . . .	361
1.1	More on Herglotz Functions . . . . .	361
1.2	The Continuous Case . . . . .	366
1.3	The Lattice Case . . . . .	373
2	Periodic and Almost Periodic Potentials . . . . .	381
2.1	Floquet Theory . . . . .	381
2.2	Inverse Spectral Theory . . . . .	391
2.3	The Lattice Case . . . . .	392
2.4	Almost Periodic Potentials . . . . .	395
3	The Absolutely Continuous Spectrum . . . . .	400
3.1	The Essential Support of the Absolutely Continuous Spectrum . . . . .	400
3.2	Support Theorems and Deterministic Potentials . . . . .	408
4	Inverse Spectral Theory . . . . .	412
4.1	The Continuous Case . . . . .	415
4.2	The Lattice Case . . . . .	420
5	Miscellaneous . . . . .	422
5.1	Potentials Taking Finitely Many Values . . . . .	422

5.2 A Remark on the Multidimensional Case . . . . .	424
6 Problems . . . . .	425
7 Notes and Complements . . . . .	432
<b>VIII LOCALIZATION IN ONE DIMENSION</b>	<b>439</b>
1 Pointwise Theory . . . . .	441
1.1 Kotani's Trick . . . . .	443
1.2 The Discrete Case . . . . .	448
1.3 The General Case . . . . .	455
2 Perturbation Theory . . . . .	457
3 Operator Theory . . . . .	461
3.1 The Discrete I.I.D. Model . . . . .	463
3.2 The Markov Model . . . . .	469
3.3 The Discrete I.I.D. Model on the Strip . . . . .	476
4 Localization for Singular Potentials . . . . .	480
5 Non-Stationary Processes . . . . .	491
5.1 The Discrete Case . . . . .	492
5.2 The Continuous Case . . . . .	495
6 Problems . . . . .	503
7 Notes and Complements . . . . .	511
<b>IX LOCALIZATION IN ANY DIMENSION</b>	<b>515</b>
1 Exponential Decay of the Green's Function at Fixed Energy . . . . .	517
1.1 Decay of the Green's Function in Boxes . . . . .	525
1.2 Decay of the Green's Function in $\mathbb{Z}^d$ . . . . .	530
2 Localization for A.C. Potentials . . . . .	535
2.1 Pointwise Theory . . . . .	535
2.2 Perturbation Theory . . . . .	537
3 A Direct Proof of Localization . . . . .	541
3.1 Examples . . . . .	542
3.2 The Proof . . . . .	544
3.3 Extensions . . . . .	547
4 Problems . . . . .	548
5 Notes and Complements . . . . .	551
<b>Bibliography</b> . . . . .	<b>557</b>
<b>Notation Index</b> . . . . .	<b>580</b>
<b>Subject Index</b> . . . . .	<b>583</b>