

---

# CONTENTS

<b>Introduction</b>	<b>1</b>
<b>1. The Energy and Weak Formulations</b>	<b>5</b>
1.1. The Classical Formulation and the Finite Difference Method	5
1.2. The Energy Formulation and Variational Finite Element Method	8
1.3. The Weak Formulation and Galerkin Finite Element Method	15
1.4. Comparison of the Three Formulations	18
1.5. Assembly by Nodes	21
1.6. Assembly by Elements	22
1.7. General Outline of FEM	25
1.8. Observations and References	29
Exercises	30
<b>2. The Finite Element Method for Two-Space-Variable Problems</b>	<b>37</b>
2.1. Triangular Elements and Linear Shape Functions	37

2.2.	The Energy Integral for a Simple Problem	43
2.3.	The Construction of the Element Matrices	47
2.4.	The Finite Element Program FEMI	53
2.5.	A More General Problem	61
2.6.	Observations and References	66
	Exercises	67
3.	<b>Assembly by Nodes and a Reduced System Matrix</b>	<b>73</b>
3.1.	Programming Assembly by Nodes	73
3.2.	Comparisons of Assembly by Nodes and by Elements	75
3.3.	A Reduced System Matrix	75
3.4.	Solution of the Algebraic Problem by Iteration	77
3.5.	Observations and References	79
	Exercises	79
4.	<b>Shape Functions</b>	<b>81</b>
4.1.	Linear Shape Functions on Tetrahedral Elements	81
4.2.	Quadratic Shape Functions on Interval Elements	85
4.3.	Quadratic Shape Functions on Triangular Elements	90
4.4.	Bilinear Shape Functions on Rectangular Elements	100
4.5.	Complete Cubic Shape Functions on Triangular Elements	105
4.6.	Observations and References	108
	Exercises	108
5.	<b>Error Estimates and Existence</b>	<b>111</b>
5.1.	Definitions of $a(u, \psi)$ , $H_0^1(0, L)$	112
5.2.	Linear Spaces of Real-Valued Functions	118
5.3.	Properties of $a(u, \psi)$	120

5.4.	Interpolation, Completeness, and Continuity of Functions in $H_0^1(0, L)$	123
5.5.	Equivalence of Classical, Energy, and Weak Formulations	130
5.6.	Error Estimates	131
5.7.	Existence	133
5.8.	Observations and References	137
	Exercises	138
<b>6.</b>	<b>Time-Dependent Problems</b>	<b>143</b>
6.1.	A Sample Problem	143
6.2.	Finite Difference Schemes	145
6.3.	Stability and the Lax Equivalence Theorem	150
6.4.	FEM for Implicit Time Discretization	164
6.5.	FEM for One Space Variable	167
6.6.	FEM for Two Space Variables	168
6.7.	Observations and References	171
	Exercises	172
<b>7.</b>	<b>Numerical Solution of Nonlinear Algebraic Systems</b>	<b>177</b>
7.1.	Motivating Examples	177
7.2.	One-Variable Methods	182
7.3.	Newton's Method for $N$ Unknowns	194
7.4.	Gauss-Seidel Variations of Newton's Method	204
7.5.	Quasi-Newton Method (Broyden)	209
7.6.	Continuation (Homotopy) Method	218
7.7.	Nonlinear Gauss-Seidel-SOR Method	224
7.8.	Observations and References	230
	Exercises	230
<b>8.</b>	<b>Applications to Nonlinear Partial Differential Equations</b>	<b>233</b>
8.1.	Comparison of Linear and Nonlinear FEM Problems	233
8.2.	Application to Nonlinear Heat Conduction	238

8.3.	Burger's Equation	249
8.4.	Incompressible Viscous Fluid Flow—Explicit Method	254
8.5.	Incompressible Viscous Fluid Flow—Implicit Method	265
8.6.	Stefan Problem	269
8.7.	Observations and References	284
	Exercises	284
9.	Variational Inequalities	287
9.1.	Obstacle Problem on a String—a Motivating Example	288
9.2.	Elliptic Variational Inequalities	293
9.3.	The Discrete Problem and an Algorithm	295
9.4.	Fluid Flow in a Porous Medium	299
9.5.	Parabolic Variational Inequalities	306
9.6.	Observations and References	311
	Exercises	311
Appendixes.	Some Nonlinear Problems and Their Computer Programs	313
A.1.	FEMI Written In Pascal: Steady-State Heat Conduction	313
A.2.	Newton's Method: Heat Flow in a Resistance Transducer	323
A.3.	Nonlinear Gauss-Seidel Method: Solidification of Water in a Channel—the Stefan Problem	330
A.4.	Variational Inequalities: Steady-State Flow in a Porous Medium—an Axisymmetric Water Filter	341
References		349
Index		351