

# Contents

|          |                                                      |           |
|----------|------------------------------------------------------|-----------|
| <b>0</b> | <b>Introduction</b>                                  | <b>1</b>  |
| <b>1</b> | <b>Moduli spaces of PEL structures</b>               | <b>15</b> |
| 1.1      | Shimura's construction . . . . .                     | 15        |
| 1.1.1    | Endomorphism rings . . . . .                         | 15        |
| 1.1.2    | Abelian varieties with given endomorphisms . . . . . | 16        |
| 1.1.3    | Arithmetic groups . . . . .                          | 18        |
| 1.1.4    | PEL structures . . . . .                             | 20        |
| 1.2      | Moduli spaces . . . . .                              | 22        |
| 1.2.1    | Moduli functors . . . . .                            | 22        |
| 1.2.2    | Examples . . . . .                                   | 22        |
| 1.2.3    | Coarse and fine moduli spaces . . . . .              | 25        |
| 1.2.4    | Moduli spaces of abelian varieties . . . . .         | 26        |
| 1.3      | Hyperbolic planes . . . . .                          | 29        |
| 1.3.1    | The rational groups . . . . .                        | 29        |
| 1.3.2    | Arithmetic groups . . . . .                          | 31        |
| 1.3.3    | Modular subvarieties . . . . .                       | 34        |
| <b>2</b> | <b>Arithmetic quotients</b>                          | <b>36</b> |
| 2.1      | Siegel modular varieties . . . . .                   | 37        |
| 2.1.1    | The groups . . . . .                                 | 37        |
| 2.1.2    | Compactifications . . . . .                          | 38        |
| 2.1.3    | Modular subvarieties . . . . .                       | 38        |
| 2.1.4    | Commensurable subgroups . . . . .                    | 39        |
| 2.2      | Picard modular varieties . . . . .                   | 43        |
| 2.2.1    | The groups . . . . .                                 | 43        |
| 2.2.2    | Compactification . . . . .                           | 45        |
| 2.2.3    | Modular subvarieties . . . . .                       | 46        |
| 2.3      | Domains of type $IV_n$ . . . . .                     | 46        |
| 2.3.1    | The groups . . . . .                                 | 46        |
| 2.3.2    | A four-dimensional family of K3-surfaces . . . . .   | 47        |
| 2.4      | Janus-like algebraic varieties . . . . .             | 50        |
| 2.4.1    | The arithmetic quotients . . . . .                   | 52        |
| 2.4.2    | The theorem . . . . .                                | 57        |

|          |                                                                       |            |
|----------|-----------------------------------------------------------------------|------------|
| 2.4.3    | A remarkable duality . . . . .                                        | 61         |
| <b>3</b> | <b>Projective embeddings</b>                                          | <b>66</b>  |
| 3.1      | The tetrahedron in $\mathbb{P}^3$ . . . . .                           | 67         |
| 3.1.1    | Arrangements defined by Weyl groups . . . . .                         | 67         |
| 3.1.2    | Rank 4 arrangements . . . . .                                         | 68         |
| 3.1.3    | The tetrahedron . . . . .                                             | 69         |
| 3.1.4    | A birational transformation . . . . .                                 | 69         |
| 3.1.5    | Fermat covers associated with arrangements . . . . .                  | 71         |
| 3.1.6    | The hypergeometric differential equation . . . . .                    | 72         |
| 3.2      | The Segre cubic $S_3$ . . . . .                                       | 74         |
| 3.2.1    | Segre's cubic primal . . . . .                                        | 74         |
| 3.2.2    | A birational transformation . . . . .                                 | 76         |
| 3.2.3    | Uniformization . . . . .                                              | 78         |
| 3.2.4    | Moduli interpretation . . . . .                                       | 83         |
| 3.3      | The Igusa quartic $\mathcal{I}_4$ . . . . .                           | 84         |
| 3.3.1    | The quartic locus associated to a configuration of 15 lines . . . . . | 84         |
| 3.3.2    | Igusa's results . . . . .                                             | 86         |
| 3.3.3    | Moduli interpretation . . . . .                                       | 89         |
| 3.3.4    | Birational transformations . . . . .                                  | 91         |
| 3.3.5    | The Siegel modular threefold of level 4 . . . . .                     | 93         |
| 3.4      | The Hessian varieties of $S_3$ and $\mathcal{I}_4$ . . . . .          | 94         |
| 3.4.1    | The Nieto quintic . . . . .                                           | 94         |
| 3.4.2    | Two birational transformations . . . . .                              | 98         |
| 3.4.3    | Moduli interpretation . . . . .                                       | 100        |
| 3.4.4    | A conjecture . . . . .                                                | 103        |
| 3.5      | The Coble variety $\mathcal{Y}$ . . . . .                             | 105        |
| <b>4</b> | <b>The 27 lines on a cubic surface</b>                                | <b>108</b> |
| 4.1      | Cubic surfaces . . . . .                                              | 109        |
| 4.1.1    | The classical approach . . . . .                                      | 109        |
| 4.1.2    | Equations . . . . .                                                   | 112        |
| 4.1.3    | Special cubic surfaces . . . . .                                      | 114        |
| 4.1.4    | Del Pezzo surfaces . . . . .                                          | 128        |
| 4.1.5    | Coble's hexahedral form . . . . .                                     | 129        |
| 4.2      | Solving algebraic equations . . . . .                                 | 133        |
| 4.2.1    | Algebraic equations . . . . .                                         | 133        |
| 4.2.2    | Resolvents . . . . .                                                  | 134        |
| 4.2.3    | The modular equation . . . . .                                        | 135        |
| 4.2.4    | Hermite, Kronecker and Brioschi . . . . .                             | 139        |
| 4.2.5    | The geometric description . . . . .                                   | 142        |
| 4.2.6    | The icosahedron . . . . .                                             | 143        |
| 4.2.7    | Klein's solution . . . . .                                            | 146        |
| 4.3      | Solving the equation of 27th degree . . . . .                         | 148        |
| 4.3.1    | The unitary reflection groups of order 25,920 . . . . .               | 150        |
| 4.3.2    | Klein's suggestion . . . . .                                          | 155        |

|          |                                                                                                  |            |
|----------|--------------------------------------------------------------------------------------------------|------------|
| 4.3.3    | Coble's solution . . . . .                                                                       | 160        |
| <b>5</b> | <b>The Burkhardt quartic</b>                                                                     | <b>168</b> |
| 5.1      | Burkhardt's quartic primal . . . . .                                                             | 169        |
| 5.2      | Subloci on $\mathcal{B}_4$ . . . . .                                                             | 172        |
| 5.2.1    | The configuration of the 45 nodes . . . . .                                                      | 172        |
| 5.2.2    | Curves on $\mathcal{B}$ . . . . .                                                                | 178        |
| 5.2.3    | Steiner primes . . . . .                                                                         | 178        |
| 5.2.4    | Jordan primes . . . . .                                                                          | 179        |
| 5.2.5    | $n$ -primes . . . . .                                                                            | 183        |
| 5.2.6    | $x$ -primes . . . . .                                                                            | 184        |
| 5.2.7    | $\mathcal{B}_4$ is rational . . . . .                                                            | 184        |
| 5.3      | $\mathcal{B}_4$ is self-Steinerian . . . . .                                                     | 185        |
| 5.3.1    | The Witting configuration in $\mathbb{P}^3$ . . . . .                                            | 187        |
| 5.3.2    | The map $\text{Hess}(\mathcal{B}) \rightarrow \mathcal{ST}(\mathcal{B}) = \mathcal{B}$ . . . . . | 190        |
| 5.3.3    | Level 3 structures on Kummer surfaces . . . . .                                                  | 190        |
| 5.4      | The solution of the Burkhardt form problem . . . . .                                             | 195        |
| 5.5      | Ball quotients . . . . .                                                                         | 198        |
| 5.5.1    | Logarithmic Chern classes . . . . .                                                              | 198        |
| 5.5.2    | Relative proportionality . . . . .                                                               | 200        |
| 5.6      | Uniformization . . . . .                                                                         | 202        |
| 5.6.1    | Step I . . . . .                                                                                 | 203        |
| 5.6.2    | Step II . . . . .                                                                                | 207        |
| 5.7      | Moduli interpretation . . . . .                                                                  | 217        |
| 5.7.1    | Abelian surfaces with a level 3 structure . . . . .                                              | 217        |
| 5.7.2    | Abelian fourfolds with complex multiplication . . . . .                                          | 218        |
| 5.7.3    | The dual variety . . . . .                                                                       | 221        |
| <b>6</b> | <b>A Gem of the modular universe</b>                                                             | <b>222</b> |
| 6.1      | The Weyl group $W(E_6)$ . . . . .                                                                | 223        |
| 6.1.1    | Notations . . . . .                                                                              | 223        |
| 6.1.2    | Roots . . . . .                                                                                  | 225        |
| 6.1.3    | Vectors . . . . .                                                                                | 226        |
| 6.1.4    | The arrangement defined by $W(E_6)$ . . . . .                                                    | 228        |
| 6.1.5    | Special Loci . . . . .                                                                           | 228        |
| 6.1.6    | The configuration defined by the 36 points . . . . .                                             | 232        |
| 6.1.7    | Invariants . . . . .                                                                             | 234        |
| 6.2      | The invariant quintic . . . . .                                                                  | 234        |
| 6.2.1    | Equation . . . . .                                                                               | 234        |
| 6.2.2    | Singular locus . . . . .                                                                         | 235        |
| 6.2.3    | Resolution of singularities . . . . .                                                            | 237        |
| 6.2.4    | $\mathcal{I}_5$ is rational . . . . .                                                            | 238        |
| 6.3      | Hyperplane sections . . . . .                                                                    | 239        |
| 6.3.1    | Lower-dimensional intersections . . . . .                                                        | 239        |
| 6.3.2    | Reducible hyperplane sections . . . . .                                                          | 239        |
| 6.3.3    | Special hyperplane sections . . . . .                                                            | 241        |

|       |                                                             |     |
|-------|-------------------------------------------------------------|-----|
| 6.3.4 | Generic hyperplane sections . . . . .                       | 241 |
| 6.3.5 | Tangent hyperplane sections . . . . .                       | 244 |
| 6.4   | Projection from a triple point . . . . .                    | 244 |
| 6.4.1 | The cuspidal model . . . . .                                | 244 |
| 6.4.2 | Projection from a triple point . . . . .                    | 246 |
| 6.4.3 | Double octics and quintic hypersurfaces . . . . .           | 247 |
| 6.4.4 | The dual picture . . . . .                                  | 249 |
| 6.5   | $\mathcal{I}_5$ and cubic surfaces . . . . .                | 251 |
| 6.5.1 | The Picard group . . . . .                                  | 251 |
| 6.5.2 | $\mathcal{I}_5$ and cubic surfaces: combinatorics . . . . . | 251 |
| 6.6   | The dual variety . . . . .                                  | 252 |
| 6.6.1 | Degree . . . . .                                            | 252 |
| 6.6.2 | Singular locus . . . . .                                    | 253 |
| 6.6.3 | Reducible hyperplane sections . . . . .                     | 254 |
| 6.6.4 | Special hyperplane sections . . . . .                       | 254 |

## Appendices 255

### A Rational groups of hermitian type 255

|       |                                                       |     |
|-------|-------------------------------------------------------|-----|
| A.1   | Algebras . . . . .                                    | 255 |
| A.1.1 | Hermitian forms over skew fields . . . . .            | 255 |
| A.1.2 | Central simple algebras . . . . .                     | 257 |
| A.1.3 | Algebras with involution . . . . .                    | 257 |
| A.1.4 | Cyclic algebras . . . . .                             | 259 |
| A.1.5 | Quaternion algebras . . . . .                         | 260 |
| A.2   | Weil's correspondence . . . . .                       | 262 |
| A.2.1 | Over $\mathbb{C}$ . . . . .                           | 262 |
| A.2.2 | Over $k$ . . . . .                                    | 263 |
| A.2.3 | Lie algebras . . . . .                                | 263 |
| A.2.4 | Over $\mathbb{R}$ . . . . .                           | 264 |
| A.3   | Classification . . . . .                              | 265 |
| A.3.1 | Split orthogonal case . . . . .                       | 266 |
| A.3.2 | Non-split orthogonal case . . . . .                   | 267 |
| A.3.3 | Split symplectic case . . . . .                       | 269 |
| A.3.4 | Non-split symplectic case . . . . .                   | 269 |
| A.3.5 | Split unitary case . . . . .                          | 270 |
| A.3.6 | Non-split unitary case . . . . .                      | 271 |
| A.4   | Rational parabolics and boundary components . . . . . | 272 |
| A.4.1 | Boundary components . . . . .                         | 273 |
| A.5   | Orders and arithmetic groups . . . . .                | 275 |
| A.5.1 | Orders in associative algebras . . . . .              | 276 |
| A.5.2 | Arithmetic groups – classical cases . . . . .         | 278 |

|          |                                                 |            |
|----------|-------------------------------------------------|------------|
| <b>B</b> | <b>Some classical algebraic geometry</b>        | <b>282</b> |
| B.1      | Invariants . . . . .                            | 282        |
| B.1.1    | Invariants and covariants . . . . .             | 282        |
| B.1.2    | Invariants of finite group actions . . . . .    | 288        |
| B.2      | Symbolic notation . . . . .                     | 290        |
| B.2.1    | Notations . . . . .                             | 291        |
| B.2.2    | The fundamental theorem . . . . .               | 293        |
| B.2.3    | Mixed concomitants . . . . .                    | 295        |
| B.2.4    | The transference principle of Clebsch . . . . . | 296        |
| B.3      | Binary forms . . . . .                          | 297        |
| B.3.1    | Invariants as functions of the roots . . . . .  | 297        |
| B.3.2    | Quintics . . . . .                              | 298        |
| B.3.3    | Sextics . . . . .                               | 300        |
| B.4      | Cubic surfaces . . . . .                        | 302        |
| B.4.1    | Invariants and linear covariants . . . . .      | 302        |
| B.4.2    | Hexahedral form . . . . .                       | 307        |
| B.5      | Some quartic surfaces . . . . .                 | 310        |
| B.5.1    | Kummer surfaces . . . . .                       | 310        |
| B.5.2    | Desmic surfaces . . . . .                       | 311        |
| B.5.3    | Symmetroids and Weddle surfaces . . . . .       | 315        |
|          | <b>References</b>                               | <b>320</b> |
|          | <b>Index</b>                                    | <b>327</b> |