

CONTENTS

1	Bernoulli, Euler and Stirling Numbers	1
1.1.	Bernoulli Numbers and Polynomials, 2	
1.1.1.	Definitions and Properties, 3	
1.1.2.	A Simple Difference Equation, 6	
1.1.3.	Euler's Summation Formula, 9	
1.2.	Euler Numbers and Polynomials, 14	
1.2.1.	Definitions and Properties, 15	
1.2.2.	Boole's Summation Formula, 17	
1.3.	Stirling Numbers, 18	
1.4.	Remarks and Comments for Further Reading, 21	
1.5.	Exercises and Further Examples, 22	
2	Useful Methods and Techniques	29
2.1.	Some Theorems from Analysis, 29	
2.2.	Asymptotic Expansions of Integrals, 31	
2.2.1.	Watson's Lemma, 32	
2.2.2.	The Saddle Point Method, 34	
2.2.3.	Other Asymptotic Methods, 38	
2.3.	Exercises and Further Examples, 39	
3	The Gamma Function	41
3.1.	Introduction, 41	
3.1.1.	The Fundamental Recursion Property, 42	
3.1.2.	Another Look at the Gamma Function, 42	
3.2.	Important Properties, 43	
3.2.1.	Prym's Decomposition, 43	
3.2.2.	The Cauchy-Saalschütz Representation, 44	

- 3.2.3. The Beta Integral, 45
- 3.2.4. The Multiplication Formula, 46
- 3.2.5. The Reflection Formula, 46
- 3.2.6. The Reciprocal Gamma Function, 48
- 3.2.7. A Complex Contour for the Beta Integral, 49
- 3.3. Infinite Products, 50
 - 3.3.1. Gauss' Multiplication Formula, 52
- 3.4. Logarithmic Derivative of the Gamma Function, 53
- 3.5. Riemann's Zeta Function, 57
- 3.6. Asymptotic Expansions, 61
 - 3.6.1. Estimations of the Remainder, 64
 - 3.6.2. Ratio of Two Gamma Functions, 66
 - 3.6.3. Application of the Saddle Point Method, 69
- 3.7. Remarks and Comments for Further Reading, 71
- 3.8. Exercises and Further Examples, 72

4 Differential Equations

79

- 4.1. Separating the Wave Equation, 79
 - 4.1.1. Separating the Variables, 81
- 4.2. Differential Equations in the Complex Plane, 83
 - 4.2.1. Singular Points, 83
 - 4.2.2. Transformation of the Point at Infinity, 84
 - 4.2.3. The Solution Near a Regular Point, 85
 - 4.2.4. Power Series Expansions Around a Regular Point, 90
 - 4.2.5. Power Series Expansions Around a Regular Singular Point, 92
- 4.3. Sturm's Comparison Theorem, 97
- 4.4. Integrals as Solutions of Differential Equations, 98
- 4.5. The Liouville Transformation, 103
- 4.6. Remarks and Comments for Further Reading, 104
- 4.7. Exercises and Further Examples, 104

5 Hypergeometric Functions

107

- 5.1. Definitions and Simple Relations, 107
- 5.2. Analytic Continuation, 109
 - 5.2.1. Three Functional Relations, 110
 - 5.2.2. A Contour Integral Representation, 111
- 5.3. The Hypergeometric Differential Equation, 112
- 5.4. The Singular Points of the Differential Equation, 114
- 5.5. The Riemann-Papperitz Equation, 116
- 5.6. Barnes' Contour Integral for $F(a, b; c; z)$, 119
- 5.7. Recurrence Relations, 121
- 5.8. Quadratic Transformations, 122
- 5.9. Generalized Hypergeometric Functions, 124
 - 5.9.1. A First Introduction to q -functions, 125

- 5.10. Remarks and Comments for Further Reading, 127
- 5.11. Exercises and Further Examples, 128

6 Orthogonal Polynomials 133

- 6.1. General Orthogonal Polynomials, 133
 - 6.1.1. Zeros of Orthogonal Polynomials, 137
 - 6.1.2. Gauss Quadrature, 138
- 6.2. Classical Orthogonal Polynomials, 141
- 6.3. Definitions by the Rodrigues Formula, 142
- 6.4. Recurrence Relations, 146
- 6.5. Differential Equations, 149
- 6.6. Explicit Representations, 151
- 6.7. Generating Functions, 154
- 6.8. Legendre Polynomials, 156
 - 6.8.1. The Norm of the Legendre Polynomials, 156
 - 6.8.2. Integral Expressions for the Legendre Polynomials, 156
 - 6.8.3. Some Bounds on Legendre Polynomials, 157
 - 6.8.4. An Asymptotic Expansion as n is Large, 158
- 6.9. Expansions in Terms of Orthogonal Polynomials, 160
 - 6.9.1. An Optimal Result in Connection with Legendre Polynomials, 160
 - 6.9.2. Numerical Aspects of Chebyshev Polynomials, 162
- 6.10. Remarks and Comments for Further Reading, 164
- 6.11. Exercises and Further Examples, 164

7 Confluent Hypergeometric Functions 171

- 7.1. The M -function, 172
- 7.2. The U -function, 175
- 7.3. Special Cases and Further Relations, 177
 - 7.3.1. Whittaker Functions, 178
 - 7.3.2. Coulomb Wave Functions, 178
 - 7.3.3. Parabolic Cylinder Functions, 179
 - 7.3.4. Error Functions, 180
 - 7.3.5. Exponential Integrals, 180
 - 7.3.6. Fresnel Integrals, 182
 - 7.3.7. Incomplete Gamma Functions, 185
 - 7.3.8. Bessel Functions, 186
 - 7.3.9. Orthogonal Polynomials, 186
- 7.4. Remarks and Comments for Further Reading, 186
- 7.5. Exercises and Further Examples, 187

8 Legendre Functions 193

- 8.1. The Legendre Differential Equation, 194
- 8.2. Ordinary Legendre Functions, 194

- 8.3. Other Solutions of the Differential Equation, 196
- 8.4. A Few More Series Expansions, 198
- 8.5. The function $Q_n(z)$, 200
- 8.6. Integral Representations, 202
- 8.7. Associated Legendre Functions, 209
- 8.8. Remarks and Comments for Further Reading, 213
- 8.9. Exercises and Further Examples, 214

9 Bessel Functions 219

- 9.1. Introduction, 219
- 9.2. Integral Representations, 220
- 9.3. Cylinder Functions, 223
- 9.4. Further Properties, 227
- 9.5. Modified Bessel Functions, 232
- 9.6. Integral Representations for the I - and K -Functions, 234
- 9.7. Asymptotic Expansions, 238
- 9.8. Zeros of Bessel Functions, 241
- 9.9. Orthogonality Relations, Fourier-Bessel Series, 244
- 9.10. Remarks and Comments for Further Reading, 247
- 9.11. Exercises and Further Examples, 247

10 Separating the Wave Equation 257

- 10.1. General Transformations, 258
- 10.2. Special Coordinate Systems, 259
 - 10.2.1. Cylindrical Coordinates, 259
 - 10.2.2. Spherical Coordinates, 261
 - 10.2.3. Elliptic Cylinder Coordinates, 263
 - 10.2.4. Parabolic Cylinder Coordinates, 264
 - 10.2.5. Oblate Spheroidal Coordinates, 266
- 10.3. Boundary Value Problems, 268
 - 10.3.1. Heat Conduction in a Cylinder, 268
 - 10.3.2. Diffraction of a Plane Wave Due to a Sphere, 270
- 10.4. Remarks and Comments for Further Reading, 271
- 10.5. Exercises and Further Examples, 272

11 Special Statistical Distribution Functions 275

- 11.1. Error Functions, 275
 - 11.1.1. The Error Function and Asymptotic Expansions, 276
- 11.2. Incomplete Gamma Functions, 277
 - 11.2.1. Series Expansions, 279
 - 11.2.2. Continued Fraction for $\Gamma(a, z)$, 280
 - 11.2.3. Contour Integral for the Incomplete Gamma Functions, 282
 - 11.2.4. Uniform Asymptotic Expansions, 283

11.2.5. Numerical Aspects, 286	
11.3. Incomplete Beta Functions, 288	
11.3.1. Recurrence Relations, 289	
11.3.2. Contour Integral for the Incomplete Beta Function, 290	
11.3.3. Asymptotic Expansions, 291	
11.3.4. Numerical Aspects, 297	
11.4. Non-Central Chi-Squared Distribution, 298	
11.4.1. A Few More Integral Representations, 300	
11.4.2. Asymptotic Expansion; m Fixed, j Large, 302	
11.4.3. Asymptotic Expansion; j Large, m Arbitrary, 303	
11.4.4. Numerical Aspects, 305	
11.5. An Incomplete Bessel Function, 308	
11.6. Remarks and Comments for Further Reading, 309	
11.7. Exercises and Further Examples, 310	
12 Elliptic Integrals and Elliptic Functions	319
12.1. Complete Integrals of the First and Second Kind, 315	
12.1.1. The Simple Pendulum, 316	
12.1.2. Arithmetic Geometric Mean, 318	
12.2. Incomplete Elliptic Integrals, 321	
12.3. Elliptic Functions and Theta Functions, 322	
12.3.1. Elliptic Functions, 323	
12.3.2. Theta Functions, 324	
12.4. Numerical Aspects, 328	
12.5. Remarks and Comments for Further Reading, 329	
12.6. Exercises and Further Examples, 330	
13 Numerical Aspects of Special Functions	333
13.1. A Simple Recurrence Relation, 334	
13.2. Introduction to the General Theory, 335	
13.3. Examples, 338	
13.4. Miller's Algorithm, 343	
13.5. How to Compute a Continued Fraction, 347	
Bibliography	349
Notations and Symbols	361
Index	365