

Contents

Preface

xiii

1	Introduction to Probability	1
1.1	Intuitive Explanation	1
1.1.1	Frequencies	1
1.1.2	Number of Favorable Cases Over The Total Number of Cases	1
1.2	Axiomatic Definition	2
1.2.1	Random Experiment	2
1.2.2	Event	2
1.2.3	Algebra of Events	3
1.2.4	Probability Axioms	3
1.2.5	Conditional Probabilities	5
1.2.6	Independent Events	6
2	Introduction to Random Variables	9
2.1	Random Variables	9
2.1.1	Cumulative Distribution Function	10
2.1.2	Probability Density Function	10
2.1.3	Mean, Variance and Higher Moments of a Random Variable	14
2.1.4	Characteristic Function of a Random Variable	19
2.2	Random vectors	21
2.2.1	Cumulative Distribution Function of a Random Vector	22
2.2.2	Probability Density Function of a Random Vector	22
2.2.3	Marginal Distribution of a Random Vector	23
2.2.4	Conditional Distribution of a Random Vector	24
2.2.5	Mean, Variance and Higher Moments of a Random Vector	26
2.2.6	Characteristic Function of a Random Vector	29
2.3	Transformation of Random Variables	30
2.4	Transformation of Random Vectors	34
2.5	Approximation of the Standard Normal Cumulative Distribution Function	36

3	Random Sequences	39
3.1	Sum of Independent Random Variables	39
3.2	Law of Large Numbers	41
3.3	Central Limit Theorem	42
3.4	Convergence of Sequences of Random Variables	44
3.4.1	Sure Convergence	44
3.4.2	Almost Sure Convergence	45
3.4.3	Convergence in Probability	45
3.4.4	Convergence in Quadratic Mean	45
4	Introduction to Computer Simulation of Random Variables	47
4.1	Uniform Random Variable Generator	48
4.2	Generating Discrete Random Variables	48
4.2.1	Finite Discrete Random Variables	48
4.2.2	Infinite Discrete Random Variables: Poisson Distribution	50
4.3	Simulation of Continuous Random Variables	51
4.3.1	Cauchy Distribution	52
4.3.2	Exponential Law	52
4.3.3	Rayleigh Random Variable	53
4.3.4	Gaussian Distribution	53
4.4	Simulation of Random Vectors	56
4.4.1	Case of a Two-Dimensional Random Vector	57
4.4.2	Cholesky Decomposition of the Variance-Covariance Matrix	57
4.4.3	Eigenvalue Decomposition of the Variance-Covariance Matrix	60
4.4.4	Simulation of a Gaussian Random Vector with MATLAB	62
4.5	Acceptance-Rejection Method	62
4.6	Markov Chain Monte Carlo Method (MCMC)	65
4.6.1	Definition of a Markov Process	65
4.6.2	Description of the MCMC Technique	65
5	Foundations of Monte Carlo Simulations	67
5.1	Basic Idea	67
5.2	Introduction to the Concept of Precision	69
5.3	Quality of Monte Carlo Simulations Results	71
5.4	Improvement of the Quality of Monte Carlo Simulations or Variance Reduction Techniques	75
5.4.1	Quadratic Resampling	75
5.4.2	Reduction of the Number of Simulations Using Antithetic Variables	77
5.4.3	Reduction of the Number of Simulations Using Control Variates	79
5.4.4	Importance Sampling	80
5.5	Application Cases of Random Variables Simulations	84
5.5.1	Application Case: Generation of Random Variables as a Function of the Number of Simulations	84
5.5.2	Application Case: Simulations and Improvement of the Simulations' Quality	87

6	Fundamentals of Quasi Monte Carlo (QMC) Simulations	91
6.1	Van Der Corput Sequence (Basic Sequence)	92
6.2	Halton Sequence	93
6.3	Faure Sequence	95
6.4	Sobol Sequence	97
6.5	Latin Hypercube Sampling	101
6.6	Comparison of the Different Sequences	102
7	Introduction to Random Processes	109
7.1	Characterization	109
7.1.1	Statistics	109
7.1.2	Stationarity	110
7.1.3	Ergodicity	111
7.2	Notion of Continuity, Differentiability and Integrability	111
7.2.1	Continuity	112
7.2.2	Differentiability	112
7.2.3	Integrability	113
7.3	Examples of Random Processes	113
7.3.1	Gaussian Process	113
7.3.2	Random Walk	114
7.3.3	Wiener Process	116
7.3.4	Brownian Bridge	118
7.3.5	Fourier Transform of a Brownian Bridge	119
7.3.6	Example of a Brownian Bridge	120
8	Solution of Stochastic Differential Equations	123
8.1	Introduction to Stochastic Calculus	124
8.2	Introduction to Stochastic Differential Equations	126
8.2.1	Ito's Integral	126
8.2.2	Ito's Lemma	126
8.2.3	Ito's Lemma in the Multi-Dimensional Case	130
8.2.4	Solutions of Some Stochastic Differential Equations	130
8.3	Introduction to Stochastic Processes with Jumps	132
8.4	Numerical Solutions of some Stochastic Differential Equations (SDE)	133
8.4.1	Ordinary Differential Equations	134
8.4.2	Stochastic Differential Equations	135
8.5	Application Case: Generation of a Stochastic Differential Equation using the Euler and Milstein Schemes	138
8.5.1	Sensitivity with Respect to the Number of Simulated Series	139
8.5.2	Sensitivity with Respect to the Confidence Interval	141
8.5.3	Sensitivity with Respect to the Number of Simulations	141
8.5.4	Sensitivity with Respect to the Time Step	142
8.6	Application Case: Simulation of a Stochastic Differential Equation with Control and Antithetic Variables	142
8.6.1	Simple Simulations	143
8.6.2	Simulations with Control Variables	143
8.6.3	Simulations with Antithetic Variables	145

8.7	Application Case: Generation of a Stochastic Differential Equation with Jumps	146
9	General Approach to the Valuation of Contingent Claims	149
9.1	The Cox, Ross and Rubinstein (1979) Binomial Model of Option Pricing	150
9.1.1	Assumptions	150
9.1.2	Price of a Call Option	151
9.1.3	Extension To N Periods	153
9.2	Black and Scholes (1973) and Merton (1973) Option Pricing Model	156
9.2.1	Fundamental Equation for the Valuation of Contingent Claims	156
9.2.2	Exact Analytical Value of European Call and Put Options	158
9.2.3	Hedging Ratios and the Sensitivity Coefficients	160
9.3	Derivation of the Black-Scholes Formula using the Risk-Neutral Valuation Principle	164
9.3.1	The Girsanov Theorem and the Risk-Neutral Probability	164
9.3.2	Derivation of the Black and Scholes Formula Under The Risk Neutralized or Equivalent Martingale Principle	165
10	Pricing Options using Monte Carlo Simulations	169
10.1	Plain Vanilla Options: European put and Call	169
10.1.1	Simple Simulations	169
10.1.2	Simulations with Antithetic Variables	171
10.1.3	Simulations with Control Variates	172
10.1.4	Simulations with Stochastic Interest Rate	177
10.1.5	Simulations with Stochastic Interest Rate and Stochastic Volatility	180
10.2	American options	182
10.2.1	Simulations Using The Least-Squares Method of Longstaff and Schwartz (2001)	183
10.2.2	Simulations Using The Dynamic Programming Technique of Barraquand and Martineau (1995)	193
10.3	Asian options	201
10.3.1	Asian Options on Arithmetic Mean	201
10.3.2	Asian Options on Geometric Mean	203
10.4	Barrier options	205
10.5	Estimation Methods for the Sensitivity Coefficients or Greeks	207
10.5.1	Pathwise Derivative Estimates	207
10.5.2	Likelihood Ratio Method	210
10.5.3	Retrieval of Volatility Method	213
11	Term Structure of Interest Rates and Interest Rate Derivatives	221
11.1	General Approach and the Vasicek (1977) Model	221
11.1.1	General Formulation	221
11.1.2	Risk Neutral Approach	224
11.1.3	Particular Case: One Factor Vasicek Model	224
11.2	The General Equilibrium Approach: The Cox, Ingersoll and Ross (CIR, 1985) model	227

11.3	The Affine Model of the Term Structure	229
11.4	Market Models	230
11.4.1	The Heath, Jarrow and Morton (HJM, 1992) Model	230
11.4.2	The Brace, Gatarek and Musiela (BGM, 1997) Model	237
12	Credit Risk and the Valuation of Corporate Securities	247
12.1	Valuation of Corporate Risky Debts: The Merton (1974) Model	247
12.1.1	The Black and Scholes (1973) Model Revisited	248
12.1.2	Application of the Model to the Valuation of a Risky Debt	249
12.1.3	Analysis of the Debt Risk	253
12.1.4	Relation Between The Firm's Asset Volatility and its Equity Volatility	256
12.2	Insuring Debt Against Default Risk	258
12.2.1	Isomorphism Between a Put Option and a Financial Guarantee	258
12.2.2	Insuring The Default Risk of a Risky Debt	260
12.2.3	Establishing a Lower Bound for the Price of the Insurance Strategy	262
12.3	Valuation of a Risky Debt: The Reduced-Form Approach	262
12.3.1	The Discrete Case with a Zero-Coupon Bond	262
12.3.2	General Case in Continuous Time	263
13	Valuation of Portfolios of Financial Guarantees	265
13.1	Valuation of a Portfolio of Loan Guarantees	265
13.1.1	Firms' and Guarantor's Dynamics	266
13.1.2	Value of Loss Per Unit of Debt	267
13.1.3	Value of Guarantee Per Unit of Debt	269
13.2	Valuation of Credit Insurance Portfolios using Monte Carlo Simulations	271
13.2.1	Stochastic Processes	272
13.2.2	Expected Shortfall and Credit Insurance Valuation	273
13.2.3	MATLAB Program	275
14	Risk Management and Value at Risk (VaR)	283
14.1	Types of Financial Risks	284
14.1.1	Market Risk	284
14.1.2	Liquidity Risk	284
14.1.3	Credit Risk	284
14.1.4	Operational Risk	284
14.2	Definition of the Value at Risk (VaR)	284
14.3	The Regulatory Environment of Basle	285
14.3.1	Stress Testing	286
14.3.2	Back Testing	286
14.4	Approaches to compute VaR	286
14.4.1	Non-Parametric Approach: Historical Simulations	287
14.4.2	Parametric Approaches	287
14.5	Computing VaR by Monte Carlo Simulations	288
14.5.1	Description of the Procedure	288
14.5.2	Application: VaR of a Simple Bank Account	288

14.5.3	Application: VaR of a Portfolio Composed of One Domestic Stock and One Foreign Stock	292
15	Value at Risk (VaR) and Principal Components Analysis (PCA)	297
15.1	Introduction to the Principal Components Analysis	297
15.1.1	Graphical Illustration	297
15.1.2	Analytical Illustration	298
15.1.3	Illustrative Example of the PCA	301
15.2	Computing the VaR of a Bond Portfolio	303
15.2.1	Sample Description and Methodology	303
15.2.2	Principal Components Analysis (PCA)	305
15.2.3	Linear Interpolation or Bootstrapping for the Intermediate Spot Rates	307
15.2.4	Computing VaR by MC and QMC Simulations	308
Appendix A:	Review of Mathematics	315
A.1	Matrices	315
A.1.1	Elementary Operations on Matrices	316
A.1.2	Vectors	317
A.1.3	Properties	317
A.1.4	Determinants of Matrices	318
A.2	Solution of a System of Linear Equations	320
A.3	Matrix Decomposition	322
A.4	Polynomial and Linear Approximation	322
A.5	Eigenvectors and Eigenvalues of a Matrix	323
Appendix B:	MATLAB® Functions	325
	References and Bibliography	327
	Index	333