
Contents

Preface	V
General notations	XV
1 Derivatives and arbitrage pricing	1
1.1 Options	1
1.1.1 Main purposes	3
1.1.2 Main problems	4
1.1.3 Rules of compounding	4
1.1.4 Arbitrage opportunities and Put-Call parity formula ..	5
1.2 Risk-neutral price and arbitrage pricing	7
1.2.1 Risk-neutral price	7
1.2.2 Risk-neutral probability	8
1.2.3 Arbitrage price	8
1.2.4 A generalization of the Put-Call parity	10
1.2.5 Incomplete markets	11
2 Discrete market models	15
2.1 Discrete markets and arbitrage strategies	15
2.1.1 Self-financing and predictable strategies	16
2.1.2 Normalized market	19
2.1.3 Arbitrage opportunities and admissible strategies	20
2.1.4 Equivalent martingale measure	21
2.1.5 Change of numeraire	24
2.2 European derivatives	26
2.2.1 Pricing in an arbitrage-free market	27
2.2.2 Completeness	30
2.2.3 Fundamental theorems of asset pricing	31
2.2.4 Markov property	34
2.3 Binomial model	35
2.3.1 Martingale measure and arbitrage price	38
2.3.2 Hedging strategies	40

2.3.3	Binomial algorithm	45
2.3.4	Calibration	50
2.3.5	Binomial model and Black-Scholes formula	53
2.3.6	Black-Scholes differential equation	60
2.4	Trinomial model	62
2.4.1	Pricing and hedging in an incomplete market	66
2.5	American derivatives	72
2.5.1	Arbitrage price	74
2.5.2	Optimal exercise strategies	80
2.5.3	Pricing and hedging algorithms	83
2.5.4	Relations with European options	88
2.5.5	Free-boundary problem for American options	90
2.5.6	American and European options in the binomial model	93
3	Continuous-time stochastic processes	97
3.1	Stochastic processes and real Brownian motion	97
3.1.1	Markov property	100
3.1.2	Brownian motion and the heat equation	102
3.2	Uniqueness	103
3.2.1	Law of a continuous process	103
3.2.2	Equivalence of processes	105
3.2.3	Modifications and indistinguishable processes	107
3.2.4	Adapted and progressively measurable processes	110
3.3	Martingales	111
3.3.1	Doob's inequality	113
3.3.2	Martingale spaces: $\mathcal{M}^2$ and $\mathcal{M}_c^2$	114
3.3.3	The usual hypotheses	117
3.3.4	Stopping times and martingales	120
3.4	Riemann-Stieltjes integral	125
3.4.1	Bounded-variation functions	127
3.4.2	Riemann-Stieltjes integral and Itô formula	131
3.4.3	Regularity of the paths of a Brownian motion	134
4	Brownian integration	139
4.1	Stochastic integral of deterministic functions	140
4.2	Stochastic integral of simple processes	141
4.3	Integral of $\mathbb{L}^2$ -processes	145
4.3.1	Itô and Riemann-Stieltjes integral	149
4.3.2	Itô integral and stopping times	151
4.3.3	Quadratic variation process	153
4.3.4	Martingales with bounded variation	156
4.3.5	Co-variation process	157
4.4	Integral of $\mathbb{L}_{\text{loc}}^2$ -processes	159
4.4.1	Local martingales	161
4.4.2	Localization and quadratic variation	163

5	Itô calculus	167
5.1	Itô processes	168
5.1.1	Itô formula for Brownian motion	169
5.1.2	General formulation	174
5.1.3	Martingales+and parabolic equations	176
5.1.4	Geometric Brownian motion	176
5.2	Multi-dimensional Itô processes	179
5.2.1	Multi-dimensional Itô formula	183
5.2.2	Correlated Brownian motion+and martingales	188
5.3	Generalized Itô formulas	191
5.3.1	Itô formula and+weak derivatives	191
5.3.2	Tanaka formula+and local times	194
5.3.3	Tanaka+formula for Itô processes	197
5.3.4	Local+time and Black-Scholes formula	198
6	Parabolic PDEs with variable coefficients: uniqueness	203
6.1	Maximum principle and Cauchy-Dirichlet problem	206
6.2	Maximum principle and Cauchy problem	208
6.3	Non-negative solutions of the Cauchy problem	213
7	Black-Scholes model	219
7.1	Self-financing strategies	220
7.2	Markovian strategies and Black-Scholes equation	222
7.3	Pricing	225
7.3.1	Dividends and time-dependent parameters	228
7.3.2	Admissibility and absence of arbitrage	229
7.3.3	Black-Scholes analysis: heuristic approaches	231
7.3.4	Market price of risk	233
7.4	Hedging	236
7.4.1	The Greeks	236
7.4.2	Robustness of the model	245
7.4.3	Gamma and Vega-hedging	246
7.5	Implied volatility	248
7.6	Asian options	252
7.6.1	Arithmetic average	253
7.6.2	Geometric average	255
8	Parabolic PDEs with variable coefficients: existence	257
8.1	Cauchy problem and fundamental solution	258
8.1.1	Levi's parametrix method	260
8.1.2	Gaussian estimates and adjoint operator	261
8.2	Obstacle problem	263
8.2.1	Strong solutions	265
8.2.2	Penalization method	268

9	Stochastic differential equations	275
9.1	Strong solutions	276
9.1.1	Uniqueness	278
9.1.2	Existence	280
9.1.3	Properties of solutions	283
9.2	Weak solutions	286
9.2.1	Tanaka's example	286
9.2.2	Existence: the martingale problem	287
9.2.3	Uniqueness	290
9.3	Maximal estimates	292
9.3.1	Maximal estimates for martingales	293
9.3.2	Maximal estimates for diffusions	296
9.4	Feynman-Kač representation formulas	298
9.4.1	Exit time from a bounded domain	300
9.4.2	Elliptic-parabolic equations and Dirichlet problem	302
9.4.3	Evolution equations and Cauchy-Dirichlet problem	307
9.4.4	Fundamental solution and transition density	308
9.4.5	Obstacle problem and optimal stopping	310
9.5	Linear equations	314
9.5.1	Kalman condition	318
9.5.2	Kolmogorov equations and Hörmander condition	323
9.5.3	Examples	326
10	Continuous market models	329
10.1	Change of measure	329
10.1.1	Exponential martingales	329
10.1.2	Girsanov's theorem	332
10.1.3	Representation of Brownian martingales	334
10.1.4	Change of drift	339
10.2	Arbitrage theory	340
10.2.1	Change of drift with correlation	343
10.2.2	Martingale measures and market prices of risk	345
10.2.3	Examples	348
10.2.4	Admissible strategies and arbitrage opportunities	352
10.2.5	Arbitrage pricing	355
10.2.6	Complete markets	357
10.2.7	Parity formulas	358
10.3	Markovian models: the PDE approach	359
10.3.1	Martingale models for the short rate	361
10.3.2	Pricing and hedging in a complete model	364
10.4	Change of numeraire	366
10.4.1	LIBOR market model	370
10.4.2	Change of numeraire for Itô processes	372
10.4.3	Pricing with stochastic interest rate	374
10.5	Diffusion-based volatility models	376

10.5.1	Local and path-dependent volatility	377
10.5.2	CEV model	379
10.5.3	Stochastic volatility and the SABR model	386
11	American options	389
11.1	Pricing and hedging in the Black-Scholes model	389
11.2	American Call and Put options in the Black-Scholes model ..	395
11.3	Pricing and hedging in a complete market	398
12	Numerical methods	403
12.1	Euler method for ordinary equations	403
12.1.1	Higher order schemes	407
12.2	Euler method for stochastic differential equations	408
12.2.1	Milstein scheme	411
12.3	Finite-difference methods for parabolic equations	412
12.3.1	Localization	413
12.3.2	θ -schemes for the Cauchy-Dirichlet problem	414
12.3.3	Free-boundary problem	419
12.4	Monte Carlo methods	420
12.4.1	Simulation	423
12.4.2	Computation of the Greeks	425
12.4.3	Error analysis	427
13	Introduction to Lévy processes	429
13.1	Beyond Brownian motion	429
13.2	Poisson process	432
13.3	Lévy processes	437
13.3.1	Infinite divisibility and characteristic function	439
13.3.2	Jump measures of compound Poisson processes	444
13.3.3	Lévy-Itô decomposition	450
13.3.4	Lévy-Khintchine representation	457
13.3.5	Cumulants and Lévy martingales	460
13.4	Examples of Lévy processes	463
13.4.1	Jump-diffusion processes	464
13.4.2	Stable processes	466
13.4.3	Tempered stable processes	469
13.4.4	Subordination	471
13.4.5	Hyperbolic processes	478
13.5	Option pricing under exponential Lévy processes	480
13.5.1	Martingale modeling in Lévy markets	480
13.5.2	Incompleteness and choice of an EMM	485
13.5.3	Esscher transform	486
13.5.4	Exotic option pricing	491
13.5.5	Beyond Lévy processes	494

14 Stochastic calculus for jump processes	497
14.1 Stochastic integrals	497
14.1.1 Predictable processes	500
14.1.2 Semimartingales	504
14.1.3 Integrals with respect to jump measures	507
14.1.4 Lévy-type stochastic integrals	511
14.2 Stochastic differentials	514
14.2.1 Itô formula for discontinuous functions	514
14.2.2 Quadratic variation	515
14.2.3 Itô formula for semimartingales	518
14.2.4 Itô formula for Lévy processes	520
14.2.5 SDEs with jumps and Itô formula	525
14.2.6 PIDEs and Feynman-Kač representation	529
14.2.7 Linear SDEs with jumps	532
14.3 Lévy models with stochastic volatility	534
14.3.1 Lévy-driven models and pricing PIDEs	534
14.3.2 Bates model	537
14.3.3 Barndorff-Nielsen and Shephard model	539
15 Fourier methods	541
15.1 Characteristic functions and branch cut	542
15.2 Integral pricing formulas	545
15.2.1 Damping method	546
15.2.2 Pricing formulas	547
15.2.3 Implementation	551
15.2.4 Choice of the damping parameter	553
15.3 Fourier-cosine series expansions	562
15.3.1 Implementation	567
16 Elements of Malliavin calculus	577
16.1 Stochastic derivative	578
16.1.1 Examples	580
16.1.2 Chain rule	582
16.2 Duality	586
16.2.1 Clark-Ocone formula	588
16.2.2 Integration by parts and computation of the Greeks	590
16.2.3 Examples	594
Appendix: a primer in probability and parabolic PDEs	599
A.1 Probability spaces	599
A.1.1 Dynkin's theorems	601
A.1.2 Distributions	605
A.1.3 Random variables	608
A.1.4 Integration	610
A.1.5 Mean and variance	612

A.1.6	σ -algebras and information	618
A.1.7	Independence	619
A.1.8	Product measure and joint distribution	622
A.1.9	Markov inequality	625
A.2	Fourier transform	626
A.3	Parabolic equations with constant coefficients	630
A.3.1	A special case	631
A.3.2	General case	636
A.3.3	Locally integrable initial datum	637
A.3.4	Non-homogeneous Cauchy problem	638
A.3.5	Adjoint operator	639
A.4	Characteristic function and normal distribution	641
A.4.1	Multi-normal distribution	643
A.5	Conditional expectation	646
A.5.1	Radon-Nikodym theorem	646
A.5.2	Conditional expectation	648
A.5.3	Conditional expectation and discrete random variables	650
A.5.4	Properties of the conditional expectation	652
A.5.5	Conditional expectation in L^2	655
A.5.6	Change of measure	656
A.6	Stochastic processes in discrete time	657
A.6.1	Doob's decomposition	659
A.6.2	Stopping times	661
A.6.3	Doob's maximal inequality	665
A.7	Convergence of random variables	669
A.7.1	Characteristic function and convergence of variables ..	670
A.7.2	Uniform integrability	674
A.8	Topologies and σ -algebras	676
A.9	Generalized derivatives	678
A.9.1	Weak derivatives in $\mathbb{R}$	678
A.9.2	Sobolev spaces and embedding theorems	681
A.9.3	Distributions	682
A.9.4	Mollifiers	687
A.10	Separation of convex sets	690
References		691
Index		713