

Contents

Volume 2

Introduction	xv
List of Symbols	xxi

IV Prediction principles

18 Survey of prediction principles using ZFC and more	461
18.1 The closed unbounded filter	461
18.2 The Diamond Principle	462
18.3 The Weak Diamond Principle	464
18.4 Surprisingly short	465
18.5 A first application of $\diamond_\lambda E$	468
18.5.1 The algebraic part for proving $\text{End}_R M = A$ assuming the Diamond Principle $\diamond_\lambda E$ and the Weak Diamond $\Phi_\lambda E$	468
18.5.2 The construction of R -modules M with $\text{End}_R M = A$ assuming $\diamond$	471
18.5.3 The existence of R -modules M with $\text{End}_R M = A$ assuming $2^{\aleph_0} < 2^{\aleph_1}$	475
19 Prediction principles in ZFC: the Black Boxes and others	478
19.1 The Easy Black Box	478
19.2 The Strong Black Box	480
19.3 The Strong Black Box for endomorphism rings	480
19.4 The Strong Black Box for ultra-cotorsion-free modules	491
19.5 The General Black Box	500
19.6 The General Box for $E(R)$ -algebras	509
19.7 The Shelah Elevator	513
19.7.1 The combinatorial part of the Elevator	513
19.8 Shelah's absolutely rigid family of trees	519
19.8.1 Rigid families of coloured trees and the first ω -Erdős cardinal	520
19.9 Absolutely rigid trees	521
19.9.1 Partition principles	521
19.9.2 The existence of large families of absolutely rigid trees	524

V Endomorphism algebras and automorphism groups

20 Realising algebras	531
20.1 Realising algebras of size $\leq 2^{\aleph_0}$	531
20.2 Realising all cotorsion-free algebras	542
20.2.1 The main realisation theorem and the Strong Black Box	544
20.2.2 The main realisation theorem and the General Black Box	552
20.2.3 Cotorsion-free modules	564
20.2.4 Almost cotorsion-free, separable, slender and $\aleph_1$ -free modules	566
20.2.5 Endomorphism rings of Butler groups	573
20.2.6 Realisation theorems for torsion and mixed modules	573
20.3 Algebras of row-and-column-finite matrices	575
21 Automorphism groups of torsion-free abelian groups	578
21.1 The Hirsch-Zassenhaus-Theorem	580
21.1.1 The six primordial groups	582
21.1.2 A crucial lemma and its consequences	588
21.2 An extension to the torsion case	591
21.3 Involutions of the group of units	593
21.4 Open problems	596
22 Modules with distinguished submodules	597
22.1 The five-submodule theorem, an easy application of the Elevator	597
22.2 The four-submodule theorem, a harder case	603
22.3 Absolutely rigid abelian groups and E -rings of size $\geq \kappa(\omega)$ do not exist	616
22.3.1 Countable groups with large endomorphism rings	616
22.4 Absolutely indecomposable R_4 -modules	619
22.4.1 The main construction	619
22.4.2 Extension to fully rigid systems	622
22.4.3 Passing to absolutely fully rigid systems of R_5 -modules	623
22.5 A discussion of representations of posets	625
22.5.1 Simson's minimal posets $\mathcal{P}_1, \dots, \mathcal{P}_{110}$ of infinite projective type	627
22.6 Open problems	634
23 R-modules and fields from modules with distinguished submodules	636
23.1 Modules: the unrestricted case and the absolute case	636
23.2 A topological realisation from Theorem 22.21	641
23.3 Appendix	645

23.4	Prescribing endomorphism monoids and automorphism groups of fields	646
23.5	Absolutely rigid fields	648
23.5.1	The tools from field theory for applications of R_κ -modules	650
23.5.2	The construction of absolutely endo-rigid fields with prescribed endomorphism monoid	651
23.5.3	Sketch of the Proof of Theorem 23.25	656
23.6	Open problems	658
24	Endomorphism algebras of $\aleph_n$-free modules	659
24.1	$\aleph_1$ -free modules of cardinality $\aleph_1$	659
24.1.1	The construction of modules	660
24.1.2	The Pigeon-hole Lemma	665
24.1.3	Comparing branching points	668
24.1.4	Proof of the theorem	672
24.2	The next step: $\aleph_n$ -free modules for any natural number $n \in \mathbb{N}$	673
24.3	The basics for the new Combinatorial Black Box	675
24.3.1	The basics for the Easy Black Box	675
24.3.2	Algebraic preliminaries for $\aleph_n$ -free modules	676
24.4	$\aleph_n$ -free A -modules	677
24.5	The triple-homomorphism ρ and freeness	682
24.6	Chains of triples	693
24.7	The Step Lemma	695
24.7.1	The case of $f = 0$	698
24.7.2	The case of $f > 0$	699
24.7.3	Preparing the predictions on G_1 for the Step Lemma	699
24.8	Application of the Strong Black Box	709
24.9	Fully rigid systems of $\aleph_k$ -free R -modules with a prescribed R -algebra	715
24.10	Open problems	716

VI Modules and rings related to algebraic topology

25	Localisations and cellular covers, the general theory for R-modules	719
25.1	A sketch of the categorical settings	719
25.1.1	Localisations represented by morphisms	721
25.2	Localisations not represented by morphisms	723
25.3	Constructing localisations of R -modules	726
25.4	Cellular covers of R -modules	733
25.5	Some additions and cosmetics on cellular covers and localisations	741

25.6	Excursion on localisations and cellular covers of non-commutative groups	743
25.6.1	Passing to localisations of groups	743
25.6.2	Passing to cellular covers of groups	745
25.7	Open problems	751
26	Tame and wild localisations of size $\leq 2^{\aleph_0}$	752
26.1	Tame localisations	752
26.2	The wild case: classical $E(R)$ -algebras	754
26.2.1	The definition and its connection to problems in algebraic topology	754
26.3	Characterizations of $E(R)$ -algebras	759
26.4	Construction of cotorsion-free $E(R)$ -algebras of rank $\leq 2^{\aleph_0}$	761
26.5	$E(R)$ -algebras and uniquely transitive modules	764
26.5.1	UT-modules over principal ideal domains	766
26.5.2	Pure-invertible algebras	767
26.5.3	The inductive step for the construction of UT-modules	768
26.5.4	The construction of UT-modules	771
27	Tame cellular covers	773
27.1	Tame cellular covers of abelian groups	773
27.1.1	Decompositions of cellular covers	773
27.2	Cellular covers of divisible groups	775
27.3	Cellular covers of torsion and mixed groups	778
27.4	When the only cellular covers are the trivial ones	780
27.5	Open problems	781
28	Wild cellular covers	782
28.1	Cellular covers of subgroups of $\mathbb{Q}$	782
28.2	The kernels of rank $< 2^{\aleph_0}$ for cellular covering maps	783
28.3	Characterizing kernels of cellular covers of abelian groups	787
29	Absolute E-rings	792
29.1	A very simple example shows the idea of the proof	793
29.2	Constructing strongly rigid coloured trees	795
29.2.1	A shift map for trees	796
29.2.2	Strongly rigid trees	799
29.3	The construction of E -rings	801
29.4	Invariant principal ideals of R	802
29.4.1	The main theorem and consequences	808
29.4.2	Large families of absolute E -rings	809
29.5	The existence of absolute E -modules	810
29.6	Open problems	812

VII Large cellular covers, localisations and $E(R)$ -algebras

30 Large kernels of cellular covers and large localisations	815
30.1 Large kernels of cellular covers	815
30.2 Large cotorsion-free $E(R)$ -algebras	819
30.3 Localisations of cotorsion-free modules	823
30.3.1 Localisations of free R -modules	823
30.3.2 Localisations of subgroups of $\mathbb{Q}$	826
30.3.3 A report on localisations of finite simple groups	830
30.3.4 Discussing $\aleph_1$ -free $E(R)$ -algebras of cardinality $\aleph_1$	830
30.4 Open problems	831
31 Mixed $E(R)$-modules over Dedekind domains	832
31.1 The construction of mixed $E(R)$ -modules	836
31.1.1 $E(R)$ -modules whose torsion part is a direct summand	836
31.1.2 $E(R)$ -modules whose torsion part is not a direct summand	837
32 $E(R)$-modules with cotorsion	840
32.1 The construction of $E(R)$ -algebras with cotorsion	847
32.2 Open problems	849

VIII Some useful classes of algebras

33 Generalised $E(R)$-algebras	853
33.1 Background, strategy, the basic setting and the main result	853
33.2 The theory of skeletons	856
33.3 Bodies	861
33.4 Types	865
33.5 Small cancellation of types	867
33.6 Arbitrarily large free skeletons	877
33.7 The algebraic structure of free bodies $\mathfrak{B}_Y$	890
33.8 The Step Lemma	892
33.9 The main construction using the diamond principle	904
33.10 Proof of the Main Theorem with $\diamond_\kappa E$	905
33.11 The main construction in ZFC	906
33.12 Proof of the Main Theorem with the General Black Box	910
33.13 Appendix: rigid systems of generalised $E(R)$ -algebras	911
33.14 Open problems	912

34 Some more useful classes of algebras	913
34.1 Leavitt type rings: the discrete case	913
34.2 Algebras with a Hausdorff topology	920
34.3 Realising particular algebras as endomorphism algebras	926
Bibliography	933
Index	965

Volume 1

Introduction	xvii
List of Symbols	xxv

I Some useful classes of modules

1 S-completions	3
1.1 Support of elements in $\hat{B}$ – a first step	10
1.1.1 Norms	11
1.2 Uncountable S in completions	12
1.3 Modules of cardinality $\leq 2^{\aleph_0}$	14
1.3.1 Algebraically independent elements	14
2 Pure-injective modules	22
2.1 Direct limits, finitely presented modules and pure submodules	22
2.2 Characterizations of pure-injective modules	37
3 Mittag-Leffler modules	47
3.1 Characterizations of Mittag-Leffler modules	47
3.2 Flat Mittag-Leffler modules	60
3.3 Unions of countable pure chains of pure-projective modules	62
3.4 Locally projective modules	65
3.5 Open problems	79
4 Slender modules	80
4.1 Factors of products and slender modules	80
4.1.1 Radicals commuting with products	87
4.1.2 Modules with a linear topology	91
4.1.3 Characterizing slender modules by excluding submodules .	100
4.2 Slender modules over Dedekind domains	104
4.3 Open problems	111

II Approximations and cotorsion pairs

5	Approximations of modules	115
5.1	Preenvelopes and precovers	115
5.2	Cotorsion pairs and Tor-pairs	120
5.3	Minimal approximations	126
5.4	Open problems	130
6	Complete cotorsion pairs	131
6.1	Ext and direct limits	131
6.2	The abundance of complete cotorsion pairs	135
6.3	Ext and inverse limits	143
6.4	Open problems	154
7	Hill lemma and its applications	155
7.1	The general version of the Hill Lemma	156
7.2	Kaplansky theorem for cotorsion pairs	161
7.3	$\mathcal{C}$ -socle sequences and $\text{Filt}(\mathcal{C})$ -precovers	164
7.4	Singular compactness for $\mathcal{C}$ -filtered modules	170
7.5	Ascending and descending properties of modules	174
7.6	The rank version of the Hill Lemma	180
7.7	Matlis cotorsion and strongly flat modules	181
7.7.1	Strongly flat modules over valuation domains	189
7.8	Open problems	195
8	Deconstruction of the roots of Ext	196
8.1	Approximations by modules of finite homological dimensions	196
8.2	Closure properties providing for deconstruction	205
8.2.1	The tilting case	205
8.2.2	The cotilting case	210
8.3	The closure of a cotorsion pair	215
9	Modules of projective dimension one	221
9.1	Structure of $\mathcal{P}_1$ and $\mathcal{W}\mathcal{I}$ for semiprime Goldie rings	221
9.2	The class $\lim \mathcal{P}_1$	224
9.3	Open problems	227
10	Kaplansky classes and abstract elementary classes	228
10.1	Kaplansky classes and deconstructibility	228
10.2	Flat Mittag-Leffler modules revisited	230
10.3	Abstract elementary classes of the roots of Ext	240
10.4	Open problems	251

11 Independence results for cotorsion pairs	253
11.1 Completeness of cotorsion pairs under the Diamond Principle	253
11.2 Uniformisation and cotorsion pairs not generated by a set	258
11.3 Open problems	265
12 The lattice of cotorsion pairs	266
12.1 Ultra-cotorsion-free modules and the Strong Black Box	266
12.2 Rational cotorsion pairs	275
12.3 Embedding posets into the lattice of cotorsion pairs	283
III Tilting and cotilting approximations	
13 Tilting approximations	295
13.1 Tilting modules	295
13.2 Classes of finite type	309
13.2.1 Deconstruction to countable type	310
13.2.2 Definability and the Mittag-Leffler condition	315
13.2.3 Finite type and resolving subcategories	320
13.3 Localisation of tilting modules	324
13.4 Product-completeness of tilting modules	325
13.5 Open problems	329
14 1-tilting modules and their applications	330
14.1 Tilting torsion classes	330
14.2 The structure of 1-tilting modules and classes over particular rings .	333
14.2.1 1-tilting classes over artin algebras	333
14.2.2 Tilting modules and classes over Prüfer domains	336
14.2.3 The case of valuation and Dedekind domains	345
14.3 Baer modules	347
14.3.1 Solution to the Kaplansky problem	350
14.3.2 Baer modules over hereditary artin algebras	351
14.4 Matlis localisations	353
14.5 Open problems	363
15 Cotilting classes	364
15.1 Cotilting classes and the classes of cofinite type	364
15.2 1-cotilting modules and cotilting torsion-free classes	372
15.3 Cotilting over Prüfer domains	376
15.4 Ext-rigid systems	378
15.5 Open problems	381

16 Tilting and cotilting classes over commutative noetherian rings	382
16.1 Cotilting classes and characteristic sequences	382
16.1.1 The one-dimensional case	382
16.1.2 The n -dimensional case	386
16.2 Tilting classes over commutative noetherian rings	392
16.3 Tilting and cotilting modules over 1-Gorenstein rings	395
16.4 Tor-pairs over hereditary rings	396
16.5 Open problems	398
17 Tilting approximations and the finitistic dimension conjectures	400
17.1 Finitistic dimension conjectures and the tilting module T_f	400
17.2 A formula for the little finitistic dimension of right artinian rings . .	407
17.3 Artinian rings with $\mathcal{P}^{<\omega}$ contravariantly finite	411
17.4 Open problems	417
Bibliography	419
Index	451