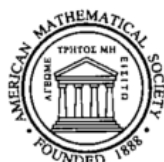


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Shock Formation in Small-Data Solutions to 3D Quasilinear Wave Equations

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