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Phase-Integral Method

Allowing Nearlying Transition Points

With adjoined papers by:

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With 17 Illustrations



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Contents

1	Phase-Integral Approximation of Arbitrary Order Generated from an Unspecified Base Function	1
1.1	Introduction	1
1.2	The So-Called WKB Approximation, Its Deficiencies in Higher Order, and Early Attempts to Remedy These Deficiencies	4
1.2.1	Derivation of the WKB Approximation	4
1.2.2	Deficiencies of the WKB Approximation in Higher Order	8
1.2.3	Phase-Integral Approximation of Arbitrary Order, Freed from the First Deficiency	11
1.3	Phase-Integral Approximation of Arbitrary Order, Generated from an Unspecified Base Function	15
1.3.1	Direct Procedure	15
1.3.2	Transformation Procedure	22
1.4	Advantage of Phase-Integral Approximation Versus WKB Approximation in Higher Order	23
1.5	Relations Between Solutions of the Schrödinger Equation and the q -Equation	26
1.5.1	Solutions of the Schrödinger Equation and Solutions of the q -Equation Expressed in Terms of Each Other	27
1.5.2	Ermakov-Lewis Invariant	29
1.6	Phase-Integral Method	30
	Appendix: Phase-Amplitude Relation	31
	References	34
2	Technique of the Comparison Equation Adapted to the Phase-Integral Method	37
2.1	Background	38
2.2	Comparison Equation Technique	42
2.2.1	Differential Equation for φ_0	46
2.2.2	Determination of the Coefficients $A_{n,0}$ and B_q	46
2.2.3	Differential Equation for φ_{2N} When $N > 0$	51
2.2.4	Regularity Properties of I_{2N} and φ_{2N} When $N > 0$	55
2.2.5	Determination of the Coefficients $A_{n,2N}$ When $N > 0$	58
2.2.6	Expressions for φ_2 and φ_4	60

2.2.7	Behavior of $\varphi_{2N}(z)$ in the Neighborhood of a First- or Second-Order Pole of $Q^2(z)$ When $N > 0$	61
2.3	Derivation of the Arbitrary-Order Phase-Integral Approximation from the Comparison Equation Solution	66
2.4	Summary of the Procedure and the Results	68
	References	70
Adjoined Papers		73
3	Problem Involving One Transition Zero	
	by Nanny Fröman and Per Olof Fröman	75
3.1	Introduction	75
3.2	Comparison Equation Solution	76
3.3	Phase-Integral Approximation Obtained from the Comparison Equation Solution	80
	References	84
4	Relations Between Different Nonoscillating Solutions of the q-Equation Close to a Transition Zero	
	by Aleksander Dzieciol, Per Olof Fröman, and Nanny Fröman	85
4.1	Introduction	85
4.2	Comparison Equation Solutions	87
4.3	Comparison Equation Expressions for Nonoscillating Solutions of the q -Equation	90
4.3.1	The Case When $\text{Re } \zeta$ Increases as z Moves Away from t in the Neighborhood of the Anti-Stokes Line A	91
4.3.2	The Case When $\text{Re } \zeta$ Decreases as z Moves Away from t in the Neighborhood of the Anti-Stokes Line A	96
4.3.3	Summary of the Results for the Two Cases in Sections 4.3.1 and 4.3.2	97
4.3.4	Application Illustrating the Consistency of the Formulas Obtained	99
4.4	Simple First-Order Formulas	100
4.5	Relations Between the α -Coefficients Associated with Different q -Functions, in Terms of Which a Given Solution $\psi(z)$ is Expressed	102
4.6	Condition for Determination of Regge Pole Positions	104
	References	108
5	Cluster of Two Simple Transitions Zeros	
	by Nanny Fröman, Per Olof Fröman, and Bengt Lundborg	109
5.1	Introduction	109
5.2	Wave Equation and Phase-Integral Approximation	111

5.3	Comparison Equation	119
5.4	Comparison Equation Solution	120
	5.4.1 Determination of $\varphi_0(z)$ and $\overline{K}_0$	121
	5.4.2 Determination of $\varphi_{2\beta}$ and $\overline{K}_{2\beta}$ for $\beta > 0$	123
5.5	Phase-Integral Solution Obtained from the Comparison Equation Solution	125
5.6	Stokes Constants	136
5.7	Application to Complex Potential Barrier	138
5.8	Application to Regge Pole Theory	139
	Appendix: Phase-Integral Solution Obtained from the Comparison Equation Solution by Straightforward Calculation	140
	References	145
6	Phase-Integral Formulas for the Regular Wave Function When There Are Turning Points Close to a Pole of the Potential	
	by Nanny Fröman, Per Olof Fröman, and Staffan Linnaeus	149
6.1	Introduction	149
6.2	Definitions and Preparatory Calculations	151
	6.2.1 Determination of φ_0 and $A_{1,0}$	154
	6.2.2 Determination of $\varphi_{2\beta}$ and $A_{1,2\beta}$ for $\beta > 0$	156
6.3	Comparison Equation Corresponding to Scattering States	162
	6.3.1 Comparison Equation Solution	162
	6.3.2 Phase-Integral Approximation Obtained from the Comparison Equation Solution	164
	6.3.3 Behavior of the Wave Function Close to the Origin	171
	6.3.4 Summary of Formulas in Section 6.3	171
6.4	Comparison Equation Corresponding to Bound States	172
	6.4.1 Quantization Condition	172
	6.4.2 Normalized Wave Function	175
	Appendix: Calculation of $q(z)$ and $\delta^{(2n+1)}$	177
	References	181
7	Normalized Wave Function of the Radial Schrödinger Equation Close to the Origin	
	by Nanny Fröman, Per Olof Fröman, Erik Walles, and Staffan Linnaeus	183
7.1	Introduction	183
7.2	$\xi_0 > 0$	187
7.3	$\xi_0 = 0, \mu_0 \neq 0$	195
7.4	Summary of the Results Obtained in the Present Chapter and Discussion of Results Obtained by Previous Authors	198

References 199

8	Phase-Amplitude Method Combined with Comparison Equation Technique Applied to an Important Special Problem	
	by Per Olof Fröman, Anders Hökback, and Nanny Fröman	201
8.1	Introduction	201
8.2	Quantization Condition	202
8.3	Solution of the Difficulty at the Origin by Means of Comparison Equation Solutions Expressed in Terms of Coulomb Wave Functions	203
8.4	Application to a Two-Dimensional Anharmonic Oscillator	206
	References	209
9	Improved Phase-Integral Treatment of the Combined Linear and Coulomb Potential	
	by Staffan Linnaeus	211
9.1	Introduction	211
9.2	Energy Levels	212
9.3	Expectation Values	215
	Appendix: Expressions for Phase-Integral Quantities in Terms of Complete Elliptic Integrals	220
	References	222
10	High-Energy Scattering from a Yukawa Potential	
	by Staffan Linnaeus	223
10.1	Introduction	223
10.2	Phase Shifts	224
10.3	Probability Density at the Origin	225
	Appendix: Numerical Solution of the Schrödinger Equation	230
	References	231
11	Probabilities for Transitions Between Bound States in a Yukawa Potential, Calculated with Comparison Equation Technique	
	by Staffan Linnaeus	233
11.1	Introduction	233
11.2	Phase-Integral Formulas	234
11.3	Comparison Equation Formulas	236
	References	240
	Author Index	243
	Subject Index	245