

Contents

Introduction	ix
Motivation	ix
Higher Ramanujan equations over $\mathbf{Z}$; Siegel case	xi
Interlude: Grothendieck's Period Conjecture	xiii
Analytic higher Ramanujan equations, periods, and transcendence	xvi
The Hilbert-Blumenthal case and an algebraic independence conjecture	xviii
Scholia	xix
Terminology and conventions	xxi
Acknowledgments	xxv
 Part 1. The arithmetic theory of the higher Ramanujan equations	 1
Chapter 1. Symplectic vector bundles over schemes	3
1.1. Symplectic vector bundles	3
1.2. Lagrangian subbundles	3
1.3. Symplectic bases	4
Chapter 2. Symplectic-Hodge bases of principally polarized abelian schemes	7
2.1. De Rham cohomology of abelian schemes	7
2.2. Symplectic form associated to a principal polarization	7
2.3. Symplectic-Hodge bases of $H_{\mathrm{dR}}^1(X/U)$	9
Chapter 3. Abelian schemes with real multiplication	11
3.1. Symplectic vector bundles with real multiplication	11
3.2. Principally polarized abelian schemes with real multiplication	12
3.3. Symplectic-Hodge bases	13
Chapter 4. The moduli stacks $\mathcal{B}_g$ and $\mathcal{B}_F$	15
4.1. The moduli stacks $\mathcal{A}_g$ and $\mathcal{A}_F$	15
4.2. Definition of the moduli stacks $\mathcal{B}_g$ and $\mathcal{B}_F$	16
4.3. Siegel parabolic subgroup and proof of Theorem 4.2.2 for $\mathcal{B}_g$	17
4.4. Proof of Theorem 4.2.2 for $\mathcal{B}_F$	19
Chapter 5. The tangent bundles of $\mathcal{B}_g$ and $\mathcal{B}_F$; higher Ramanujan vector fields	21
5.1. Horizontal subbundles and linear connections	21
5.2. The Ramanujan subbundle $\mathcal{R}_g \subset T_{\mathcal{B}_g/\mathbf{Z}}$	22
5.3. The Ramanujan subbundle $\mathcal{R}_F \subset T_{\mathcal{B}_F/\mathbf{Z}}$	25
5.4. Recollections on the Kodaira-Spencer morphism	26

5.5.	The Kodaira-Spencer isomorphism for $\mathcal{A}_g$ and $\mathcal{A}_F$	28
5.6.	The higher Ramanujan vector fields on $\mathcal{B}_g$	29
5.7.	The higher Ramanujan vector fields on $\mathcal{B}_F$	31
Chapter 6.	Integral solution of the higher Ramanujan equations	33
6.1.	Higher Ramanujan equations over $\mathcal{B}_g$	33
6.2.	Integral solution of the higher Ramanujan equations; Siegel case	34
6.3.	Higher Ramanujan equations over $\mathcal{B}_F$	35
6.4.	Integral solution of the higher Ramanujan equations; Hilbert-Blumenthal case	36
Chapter 7.	Representability of $\mathcal{B}_g$ and $\mathcal{B}_F$ by a scheme	39
7.1.	Representability by an algebraic space	39
7.2.	Representability of $\mathcal{B}_{g, \mathbf{Z}[1/2]}$ by a quasi-projective scheme B_g	41
7.3.	B_g is quasi-affine over $\mathbf{Z}[1/2]$	43
Chapter 8.	The case of elliptic curves: explicit equations	45
8.1.	Explicit equation for the universal elliptic curve X_1 over B_1 and its universal symplectic-Hodge basis	45
8.2.	Explicit formulas for the Ramanujan vector field	47
8.3.	Explicit formulas for $\hat{\varphi}_1$	48
Part 2.	The analytic higher Ramanujan equations and periods of abelian varieties	51
Chapter 9.	Analytic families of complex tori, abelian varieties, and their uniformization	53
9.1.	Relative complex tori	53
9.2.	Riemann forms and principally polarized complex tori	54
9.3.	The category $\mathcal{A}_g^{\text{an}}$ of principally polarized complex tori of relative dimension g	55
9.4.	De Rham cohomology of complex tori	56
9.5.	Relative uniformization of complex abelian schemes	58
9.6.	Principally polarized complex tori with real multiplication	60
Chapter 10.	Analytic moduli spaces of complex abelian varieties with a symplectic-Hodge basis	63
10.1.	Descent of principally polarized complex tori	63
10.2.	Integral symplectic bases over principally polarized complex tori	64
10.3.	Principal (symplectic) level structures	65
10.4.	Symplectic-Hodge bases over complex tori	67
10.5.	The Hilbert-Blumenthal case	69
Chapter 11.	The analytic higher Ramanujan equations	73
11.1.	Definition of φ_g and statement of our main theorem in the Siegel case	73
11.2.	Preliminary results	74
11.3.	Proof of Theorem 11.1.2	76
11.4.	Compatibility of φ_g with $\hat{\varphi}_g$	77
11.5.	Analytic higher Ramanujan equations over $\mathcal{B}_F$	78
11.6.	Compatibility of φ_F and $\hat{\varphi}_F$	81

Chapter 12. Values of φ_g and φ_F ; periods of abelian varieties	83
12.1. Fields of periods of abelian varieties and statement of our main theorems	83
12.2. Period matrices	84
12.3. Auxiliary lemmas	85
12.4. Proof of Theorem 12.1.3	86
12.5. Periods of abelian varieties with real multiplication	87
Chapter 13. An algebraic independence conjecture on the values of φ_F	91
13.1. Hirzebruch-Zagier divisors and statement of the conjecture	91
13.2. Periods in the presence of complex multiplication	92
13.3. Grothendieck's Period Conjecture for abelian surfaces with real multiplication	94
Chapter 14. Group-theoretic description of the higher Ramanujan vector fields	97
14.1. Realization of $B_g(\mathbf{C})$ as an open submanifold of $\mathrm{Sp}_{2g}(\mathbf{Z}) \backslash \mathrm{Sp}_{2g}(\mathbf{C})$	97
14.2. Explicit analytic description of the higher Ramanujan vector fields v_{ij} and of φ_g	100
14.3. Group-theoretic description of $\mathcal{B}_F$, v_F , and φ_F	102
Chapter 15. Zariski-density of leaves of the higher Ramanujan foliation	107
15.1. Characterization of the leaves of the higher Ramanujan foliation	107
15.2. Auxiliary results	110
15.3. Statement and proof of our Zariski-density results	112
15.4. Derivatives of modular functions and B_g	113
Appendix A. Gauss-Manin connection on some families of elliptic curves	115
A.1. The Weierstrass elliptic curve	115
A.2. The elliptic curve $X_{/B}$ over $\mathbf{Z}[1/6]$	116
A.3. The universal elliptic curve X_{1/B_1} over $\mathbf{Z}[1/2]$	117
Bibliography	119
Index of notation	123