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Symmetry and Symmetry Breaking in Chemistry

Molecular Structures and Chemical Reactions

DE GRUYTER

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