

Lorenz J. Halbeisen

Combinatorial Set Theory

With a Gentle Introduction to Forcing

Third Edition



Springer

Contents

Part I Preliminary

1	The Setting	3
	What is Infinitary Combinatorics?	3
	The Axiom of Choice	4
	Cardinal Characteristics	5
	The Forcing Technique	6
	Ramsey Theory	6
	Notes	8
	<i>References</i>	9
2	First-Order Logic in a Nutshell	11
	Syntax: The Grammar of Symbols	11
	Semantics: Making Sense of the Symbols	18
	The Completeness Theorem	20
	Limits of First-Order Logic	22
	Notes	24
	<i>References</i>	26
3	Axioms of Set Theory	29
	Why Axioms?	29
	The Axioms of Zermelo–Fraenkel Set Theory	31
	On the Consistency of ZF	47
	Models of ZF	49
	Cardinals and Ordinals in ZF	52
	Zermelo’s Axiom of Choice	59
	Gödel’s Model of ZFC	61
	Cardinal Arithmetic in ZFC	62
	Notes	69
	<i>References</i>	76

Part II Topics in Combinatorial Set Theory

4	Overture: Ramsey's Theorem	83
	The Nucleus of Ramsey Theory	83
	Corollaries of Ramsey's Theorem	86
	Generalisations of Ramsey's Theorem	88
	Notes	92
	Related Results	93
	References	96
5	Cardinal Relations in ZF Only	99
	Basic Cardinal Relations	100
	On the Cardinals $2^{\aleph_0}$ and $\aleph_1$	105
	Ordinal Numbers Revisited	107
	More Cardinal Relations	112
	Notes	124
	Related Results	125
	References	128
6	Forms of Choice	131
	Equivalent Forms of the Axiom of Choice	131
	Weaker Forms of the Axiom of Choice	141
	Notes	157
	Related Results	159
	References	167
7	How to Make Two Balls from One	173
	Equidecomposability	173
	Hausdorff's Paradox	174
	Robinson's Decomposition	177
	Notes	184
	Related Results	184
	References	185
8	Models of Set Theory with Atoms	187
	Permutation Models	188
	Fraenkel's Permutation Models	195
	The Ordered Mostowski Model	198
	Shelah-Type Permutation Models	209
	Notes	216
	Related Results	216
	References	219
9	Thirteen Cardinals and Their Relations	221
	The Cardinals ω_1 and $\mathfrak{c}$	222
	The Cardinal $\mathfrak{p}$	222
	The Cardinals $\mathfrak{b}$ and $\mathfrak{d}$	223
	The Cardinals $\mathfrak{s}$ and $\mathfrak{t}$	224

	The Cardinals $\mathfrak{a}$ and $\mathfrak{u}$	226
	The Cardinal $\mathfrak{i}$	228
	The Cardinals $\mathfrak{par}$ and $\mathfrak{hom}$	232
	The Cardinal $\mathfrak{h}$	234
	Summary	237
	Notes	238
	Related Results	238
	<i>References</i>	243
10	The Shattering Number Revisited	247
	The Ramsey Property	247
	The Ideal of Ramsey-Null Sets	249
	The Ellentuck Topology	250
	A Generalised Suslin Operation	254
	Notes	257
	Related Results	257
	<i>References</i>	258
11	Happy Families and Their Relatives	261
	Happy Families	261
	Ramsey Ultrafilters	265
	P -points and Q -points	268
	Ramsey Families and P -families	273
	The Rudin–Keisler Ordering of Ultrafilters over ω	279
	Notes	291
	Related Results	292
	<i>References</i>	297
12	Coda: A Dual Form of Ramsey’s Theorem	299
	The Hales–Jewett Theorem	299
	Families of Partitions	304
	Carlson’s Lemma and the Partition Ramsey Theorem	307
	A Weak Form of the Halpern–Läuchli Theorem	313
	Notes	315
	Related Results	316
	<i>References</i>	320
Part III From Martin’s Axiom to Cohen’s Forcing		
13	The Idea of Forcing	325
14	Martin’s Axiom	329
	Filters on Partially Ordered Sets	329
	Weaker Forms of MA and some Consequences	333
	Notes	342
	Related Results	342
	<i>References</i>	343

15	The Notion of Forcing	345
	The Language of Forcing	345
	Generic Extensions	350
	ZFC in Generic Models	355
	Independence of CH: The Gentle Way	371
	On the Existence of Generic Filters	375
	Notes	376
	Related Results	377
	<i>References</i>	378
16	Proving Unprovability	381
	Basic Model-Theoretical Facts	381
	The Reflection Principle	383
	Countable Transitive Models of Finite Fragments of ZFC	385
	Consistency and Independence Proofs: The Proper Way	388
	Notes	391
	Related Results	392
	<i>References</i>	393
17	Models in Which AC Fails	395
	Symmetric Submodels of Generic Extensions	395
	Examples of Symmetric Models	398
	Simulating Permutation Models by Symmetric Models	406
	Notes	411
	Related Results	412
	<i>References</i>	413
18	Combining Forcing Notions	415
	Products	416
	General Products of Forcing Notions	416
	Products of Cohen Forcing	417
	A Model in Which $\mathfrak{a} < \mathfrak{c}$	419
	Iterations	421
	Two-Step Iterations	421
	General Iterations	427
	Quotient Forcing	429
	Properties of Iterated Forcing	433
	A Model in Which $\mathfrak{i} < \mathfrak{c}$	436
	Notes	439
	Related Results	439
	<i>References</i>	441
19	Models in Which $\mathfrak{p} = \mathfrak{c}$	443
	A Model in Which $\mathfrak{p} = \mathfrak{c} = \omega_2$	443
	On the Consistency of $\text{MA} + \neg\text{CH}$	445
	$\mathfrak{p} = \mathfrak{c}$ Is Preserved Under Adding a Cohen Real	446
	Notes	449

Related Results	450
<i>References</i>	450
20 Suslin's Problem	451
A Topological Characterisation of the Real Line	451
Suslin Lines and Suslin Trees	453
There May Be No Suslin Line	456
There May Be a Suslin Line	457
Notes	462
Related Results	463
<i>References</i>	463

Part IV Combinatorics of Forcing Extensions

21 Properties of Forcing Extensions	467
Dominating, Splitting, Bounded, and Unbounded Reals	467
The Laver Property and Not Adding Cohen Reals	469
The Notion of Properness	470
Preservation Theorems for Proper Forcing Notions	475
Preservation of Properness	475
Preserving ω_1 and Increasing $\mathfrak{c}$	479
Preservation of ${}^\omega\omega$ -Boundedness	484
Preservation of P -Points and Ramsey Ultrafilters	491
Notes	501
Related Results	502
<i>References</i>	503
22 Cohen Forcing Revisited	505
Properties of Cohen Forcing	505
Cohen Forcing Adds Unbounded but no Dominating Reals	505
Cohen Forcing Adds Splitting Reals	506
Cohen Reals and the Covering Number of Meagre Sets	506
A Model in Which $\mathfrak{a} < \mathfrak{d} = \mathfrak{r} = \text{cov}(\mathcal{M})$	511
A Model in Which $\mathfrak{s} = \mathfrak{b} < \mathfrak{d}$	512
Notes	514
Related Results	514
<i>References</i>	516
23 Sacks Forcing	519
Properties of Sacks Forcing	519
Sacks Forcing Is Proper and ${}^\omega\omega$ -Bounding	519
Finite Products of Sacks Forcing Do not Add Splitting Reals	521
Sacks Forcing Preserves Ramsey Ultrafilters	523
A Model with Exactly $\mathfrak{c}$ Ramsey Ultrafilters	525
Notes	527
Related Results	527
<i>References</i>	528

24	Silver-Like Forcing Notions	531
	Properties of Silver-Like Forcing	532
	Silver-Like Forcing Is Proper and ${}^\omega\omega$ -Bounding	532
	Silver-Like Forcing Adds Splitting Reals	533
	A Model in Which $\mathfrak{d} < \mathfrak{r}$	534
	Notes	535
	Related Results	535
	References	535
25	Miller Forcing	537
	Properties of Miller Forcing	539
	Miller Forcing Is Proper and Adds Unbounded Reals	539
	Miller Forcing Does not Add Splitting Reals	540
	Miller Forcing Preserves P -Points	543
	A Model in Which $\mathfrak{u} < \mathfrak{d}$	546
	Notes	546
	Related Results	546
	References	547
26	Mathias Forcing	549
	Properties of Mathias Forcing	549
	Mathias Forcing Adds Dominating Reals	549
	Mathias Forcing Has Pure Decision	550
	Mathias Forcing Is Proper and Has the Laver Property	552
	A Model in Which $\mathfrak{p} < \mathfrak{h}$	555
	There May Be a Ramsey Ultrafilter and $\text{cov}(\mathcal{M}) < \mathfrak{h}$	557
	Notes	559
	Related Results	559
	References	562
27	Laver Forcing	565
	Properties of Laver Forcing	566
	Laver Forcing Restricted to a Free Filter Is σ -centred	566
	Laver Forcing Adds Dominating Reals	566
	Laver Forcing Is Proper	567
	$\mathbb{L}_{\mathcal{U}}$ and $\mathbb{M}_{\mathcal{U}}$ for Ramsey Ultrafilters $\mathcal{U}$	570
	Finite Support Iterations of Restricted Laver Forcing	571
	A Model in Which $\mathfrak{hom} < \mathfrak{c}$	571
	A Model in Which $\mathfrak{s} < \mathfrak{b}$	574
	Notes	579
	Related Results	579
	References	581
28	How Many Ramsey Ultrafilters Exist?	583
	Properties of Shelah's Product Tree Forcing	585
	Shelah's Product Tree Forcing Is Proper and ${}^\omega\omega$ -Bounding	585
	A Further Property of Shelah's Product Tree Forcing	587

There May Exist Exactly Twenty-Seven Ramsey Ultrafilters 588

Notes 593

Related Results 594

References 594

29 Suite 595

Prelude 595

Allemande 596

Courante 598

Sarabande 600

Gavotte I & II 600

Gigue 601

References 602

Index 603

Symbols 603

Names 605

Subjects 610